Comprehensive Exam - Analysis and Topology

Tuesday, 29 May 2012:   2:00pm – 5:00pm

Examiners: Spiro Karigiannis and Laurent Marcoux

Attempt all the questions. In order to pass the examination, competence must be demonstrated in all areas.

[1] [a] (5 marks) Show that there exists a bijection from $\mathcal{P}(\mathbb{R})$, the power set of $\mathbb{R}$, onto the set of all real-valued functions on $\mathbb{R}$.

[b] (5 marks) Let $\mathcal{C}(\mathbb{R},\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous}\}$. Find a cardinal number α so that $|\mathcal{C}(\mathbb{R},\mathbb{R})| = 2^\alpha$.

[2] Let $\mathcal{H}$ be a complex Hilbert space. Given a non-empty subset $E \subseteq \mathcal{H}$, define $E^\perp = \{v \in \mathcal{H};\ \langle v, w\rangle = 0 \text{ for all } w \in E\}$.

[a] (10 marks) Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$.

(i) Given $x \in \mathcal{H}$, show that there exists $m_0 \in \mathcal{M}$ so that $\|x - m_0\| = \operatorname{dist}(x, \mathcal{M})$, where $\operatorname{dist}(x, \mathcal{M}) := \inf\{\|x - m\| : m \in \mathcal{M}\}$.

(ii) Show that with x and m_0 as in (i), $x - m_0$ is orthogonal to $\mathcal{M}$.

(iii) Conclude that $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.

[b] (5 marks) Prove that a non-empty subset $E \subseteq \mathcal{H}$ satisfies $E = (E^\perp)^\perp$ if and only if E is a closed subspace of $\mathcal{H}$.

[3] (5 marks) Let $h \in C[0,1]$. Show that every $f \in C[0,1]$ is a uniform limit of polynomials in h if and only if h is strictly monotone.

[4] (10 marks) Prove that the following limit exists, and calculate its value.

$$\lim_{n\to\infty} \int_0^\infty \left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k!)} \right) e^{-2x} dx$$

[5] Recall that $\ell_2 = \{\mathbf{x} = (x_k)_{k=1}^\infty : x_k \in \mathbb{R}\ \forall\ k \geq 1 \text{ and } \|\mathbf{x}\|_2 := (\sum_{k=1}^\infty {x_k}^2)^{1/2} < \infty\}$. Consider the following subset of ℓ_2:

$$H := \{\mathbf{x} = (x_k)_{k=1}^\infty \in \ell_2 : |x_k| \leq 1/k \text{ for all } k \geq 1\}.$$

[a] (5 marks) Consider a sequence $(\mathbf{x}_n)_{n=1}^\infty$ in H, where each $\mathbf{x}_n = (x_{n,1}, x_{n,2}, x_{n,3}, ...)$. Prove that the sequence $(\mathbf{x}_n)_{n=1}^\infty$ converges in ℓ_2 if and only if for each $k \geq 1$, the sequence $(x_{n,k})_{n=1}^\infty$ converges.

[b] (5 marks) Prove that H is compact and nowhere dense in ℓ_2.

[6] (5 marks) Let $f : [0, \infty) \to \mathbb{R}$ be continuous. Suppose that for all $x \in [0, 1]$,

$$\lim_{n\to\infty} f(nx) = 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.

Hint: Consider $A_{n,\varepsilon} := \{x \in [0, 1] : |f(kx)| \leq \varepsilon \text{ for all } k \geq n\}$.

[7] Let m denote Lebesgue measure on $[0, 1]$. Suppose $(f_n)_{n=1}^{\infty}$ and f are real-valued, Lebesgue measurable functions on $[0, 1]$ and that $(f_n)_{n=1}^{\infty}$ converges pointwise a.e. to f.

[a] (5 marks) Prove that for each pair $\varepsilon, \delta > 0$ there exist a Lebesgue measurable set $A \subseteq [0, 1]$ and an integer k such that $m([0, 1] \setminus A) < \varepsilon$ and

$$|f_n(x) - f(x)| < \delta$$

for all $x \in A$ and $n \geq k$.

[b] (5 marks) Use this to prove that for each $\varepsilon > 0$ there exists a set $B \subseteq [0, 1]$ such that $m([0, 1] \setminus B) < \varepsilon$ and $(f_n)_{n=1}^{\infty}$ converges uniformly to f on B. [This is Egoroff's Theorem.]

[c] (5 marks) Does Egoroff's Theorem hold if we replace $[0, 1]$ by $\mathbb{R}$? Justify your answer.

[8] Let Ω be a connected open set in $\mathbb{R}^2 \cong \mathbb{C}$. Recall that if $u : \Omega \to \mathbb{R}$ is a C^2 function, we say that it is harmonic if $u_{xx} + u_{yy} = 0$.

[a] (5 marks)

(i) Show that the real and imaginary parts of a holomorphic function are harmonic.

(ii) If $u : \Omega \to \mathbb{R}$, then a function $v : \Omega \to \mathbb{R}$ is a conjugate of u if $f = u + iv$ is holomorphic on Ω. Show that if u is harmonic, then $-u_y$ is a conjugate of u_x.

[b] (5 marks) Prove that a harmonic function u admits a conjugate if and only if the holomorphic function $g = u_x - iu_y$ has a primitive f in Ω. Under what topological conditions on Ω is this guaranteed to hold?

[9] (5 marks) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with power series

$$f(z) = \sum_{n=0}^{\infty} c_{a,n}(z - a)^n$$

about the point a. Suppose that for every $a \in \mathbb{C}$, at least one coefficient $c_{a,n}$ is zero. Prove that f is a polynomial.

[10] (5 marks) Let g be an entire function. Show that if g is not a polynomial, then there exists a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$ with $\lim_{n\to\infty} |z_n| = \infty$ and $\lim_{n\to\infty} g(z_n) = 0$.