Instructions: There are 10 problems on this exam, worth a total of 110 points, on the topics of real analysis, measure theory, complex analysis and topology. Attempt every problem.

1. (a) Give the definition of a $G_\delta$-set in a topological space.
   (b) Determine if the set of rational numbers $\mathbb{Q}$ is a $G_\delta$-set in $\mathbb{R}$.

2. Use contour integration to evaluate the improper integral

$$\int_0^\infty \frac{x^2}{(x^2 + 1)^2} \, dx.$$  

Make sure to justify your steps.

3. (a) Let $f = u + iv$ and $g = p + iq$ be analytic functions defined in a neighbourhood $U$ of the origin in the complex plane $\mathbb{C}$. Assume that $|g'(0)| < |f'(0)|$. Prove that there is a neighbourhood $V \subset U$ of the origin in which the function $h = f + g$ is one-to-one.
   (b) Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function which satisfies $|f(z)| \leq C(1 + |z|)^n$ for $z$ in $\mathbb{C}$, where $C > 0$ is a constant. Show that $f$ is a polynomial of degree $\leq n$.

4. (a) Prove that every compact subset of a Hausdorff space is closed.
   (b) Let $f : X \to Y$ be a bijective continuous function. If $X$ is compact and $Y$ is Hausdorff, then prove that $f$ is a homeomorphism.
   (c) Suppose $X$ is a dense subset of a topological space $Y$. If $X$ is Hausdorff, must the same be true of $Y$? Prove or give a counter-example.
5. Consider the Cantor set

\[ C = \left\{ \sum_{k=1}^{\infty} \frac{t_k}{3^k} : t_1, t_2, \ldots \in \{0, 2\} \right\} \subset [0, 1] \]

with the usual topology.

(a) Show that \( C \) is homeomorphic to \( C \times C \). [You may wish to use the product space \( P = \{0, 1\}^\mathbb{N} \).

(b) Define \( \psi : C \to [0, 1] \) by \( \psi \left( \sum_{k=1}^{\infty} \frac{t_k}{3^k} \right) = \sum_{k=1}^{\infty} \frac{t_k}{2^{k+1}} \).

(i) Show that \( \psi \) is continuous and surjective.

(ii) Is \( \psi \) a homeomorphism? Is it possible to find a homeomorphism \( \varphi : C \to [0, 1] \)? Justify your answer.

(iii) Determine if \( \psi \) satisfies the property of absolute continuity, as defined below.

Given \( \varepsilon > 0 \), there is \( \delta > 0 \) so that for any \( n \), if \( a_1 < b_1 < \ldots < a_n < b_n \) in \( C \) satisfies \( \sum_{j=1}^{n} (b_j - a_j) < \delta \), we have \( \sum_{j=1}^{n} |\psi(b_j) - \psi(a_j)| < \varepsilon \).

6. For each \( n \), let \( f_n : [0, 1] \to \mathbb{R} \) be a continuous function which satisfies that \( f_n(0) = 0 \), and \( f_n \) is continuously differentiable on \( (0, 1) \) with \( |f'_n(x)| \leq x \) for \( x \) in \( (0, 1) \).

(a) Prove that there exists a subsequence of \( (f_n)_{n=1}^{\infty} \) which converges uniformly to a continuous function \( f \).

(b) Must the limit function \( f \), in (a) above, be differentiable on \( (0, 1) \)? Prove, or provide a counter-example.

7. (a) Evaluate \( \lim_{n \to \infty} \int_{0}^{3/n} n \cos(t^3) \, dt \). Justify all of your steps.

(b) Let \( g_n : [0, 1] \to [0, \infty), n = 1, 2, \ldots \) be a sequence of measurable functions such that \( \lim_{n \to \infty} \int_{0}^{1} g_n = 0 \). Is it the case that \( \lim_{n \to \infty} g_n(t) = 0 \) for almost every \( t \) in \([0, 1]\)? Prove, or provide a counter-example.
8. Let \((X, d)\) be a metric space. For any subset \(A \subset X\) and any real number \(\epsilon > 0\), let

\[
B(A, \epsilon) = \{x \in X \mid d(x, a) < \epsilon \text{ for some } a \in A\},
\]

\[
\overline{B}(A, \epsilon) = \{x \in X \mid d(x, a) \leq \epsilon \text{ for some } a \in A\}.
\]

(a) Prove that \(B(A, \epsilon)\) is an open subset of \(X\).
(b) Prove that if \(C\) is compact then \(\overline{B}(C, \epsilon)\) is a closed subset of \(X\).
(c) Does the result in (b), above, remain true if \(C\) is not assumed to be compact? Prove, or provide a counter-example.
(d) Show that if \(C\) is compact, \(A\) is closed and \(C \cap A = \emptyset\), then there is \(\epsilon > 0\) for which \(\overline{B}(C, \epsilon) \cap \overline{B}(A, \epsilon) = \emptyset\).

9. (a) State the Stone-Weierstrass Theorem for \(\mathbb{C}\)-valued continuous functions on a compact metric space.

(b) Given a Lebesgue measurable set \(A \subset [0, 1]\), \(1 \leq p < \infty\), and \(\epsilon > 0\), show that there exists a continuous function \(g : [0, 1] \rightarrow \mathbb{R}\) such that

\[
\|\chi_A - g\|_p = \left(\int_0^1 |\chi_A - g|^p \right)^{1/p} < \epsilon.
\]

[For the case where \(A\) is a finite union of open intervals, a labelled picture will suffice.]

(c) Suppose \(f : [0, 1] \rightarrow \mathbb{R}\) is a bounded Lebesgue measurable function for which \(\int_0^1 f(t) e^{nt} dt = 0\) for \(n = 0, 1, 2, \ldots\). Show that \(f(t) = 0\) for almost every \(t\) in \([0, 1]\).

10. Let \(f(z) = 3z^{100} - e^z\).

(a) Counting multiplicities, how many zeros \(z\) does \(f\) have inside the unit circle, i.e. with \(|z| < 1\)?
(b) How many zeros of \(f\) inside the unit circle are simple?