Analysis Comprehensive Exam

May 23, 2013, MC5046, 9:00am-12:00pm

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- Attempt all questions.
- You must show all of your reasoning.

1. Let ℓ^∞ denote the space of bounded $\mathbb{R}$-valued sequences with norm $\|(x_n)\|_\infty = \sup_n |x_n|$.

   (a) Show that ℓ^∞ is a complete metric space, and that it is non-separable.

   (b) Show that $B = \{(x_n) \in \ell^\infty : \|(x_n)\|_\infty \leq 1\}$ is not compact.

2. A set $E \subset \mathbb{R}^n$ is called *midpoint convex* if, for any x, y in E we have $\frac{1}{2}(x+y) \in E$ as well.

   (a) Give an example of a set $E \subset \mathbb{R}^n$ which is midpoint convex, but not convex.

   (b) Suppose $E \subset \mathbb{R}^n$ is closed. Show that if E is midpoint convex, then it is convex.

   (c) Suppose $E \subset \mathbb{R}^n$ is open. Show that if E is midpoint convex, then it is convex.

3. Show that any non-empty open set in a separable metric space (X, d) is the union of a countable family of open balls.

4. Let $C[0,1]$ denote the space of all continuous $\mathbb{C}$-valued functions on the interval $[0,1]$ with uniform norm. Let A be the subset of all polynomials with $p(0) = p(1)$.

   (a) Prove that A is dense in $\{f \in C[0,1] : f(0) = f(1)\}$.

   (b) Let $f(t) = |t - \frac{1}{2}|$. Show that any sequence (p_n) of elements of A, converging uniformly to f, necessarily has $\lim_{n\to\infty} \deg p_n = \infty$. Here $\deg p$ is the degree of the polynomial p.

5. Let m be the Lebesgue measure and $L^1[0,1] = L^1([0,1], m)$.

(a) Let $\varepsilon > 0$ and $f \in L^1[0,1]$. Prove that there exists $\delta > 0$ such that $\int_A |f|\, dm < \varepsilon$ whenever A is measurable with $m(A) < \delta$.

(b) Suppose $f_n, f \in L^1[0,1]$, $f_n \geq 0$, $f_n \to f$ pointwise, and $\int_{[0,1]} f_n \to \int_{[0,1]} f$. Prove that $\int_E f_n \to \int_E f$ for each measurable $E \subseteq [0,1]$. [Hint: Egoroff's Theorem.]

6. How many roots does the function $g(z) = 4z^7 + 7z^4 + 1$ have within the circle $|z| = 1$?

7. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function with $|f(z)| \leq \sqrt{|z|}$ for all $|z| > R$, for some fixed $R > 0$. Prove that f is a constant.

8. Evaluate each of the following integrals.

(a) $\displaystyle\int_{|z|=2} \frac{e^z}{z^2 - 2}\, dz$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 2x + 4}\, dx$

9. (a) Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function with $f(z+1) = f(z) = f(z+i)$ for each z. Show that f is necessarily constant.

(b) Let

$$g(z) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \frac{1}{(z - n - mi)^4}.$$

Show that g defines an analytic function on $\mathbb{C} \setminus \{a + bi : a, b \in \mathbb{Z}\}$, with $g(z+1) = g(z) = g(z+i)$ for each z in its domain.

10. Let A and B be non-empty sets. We say that A has *cardinality greater than* B if there is an injection from B into A, but no bijection.

(a) Show that if A has cardinality greater than B, and B has cardinality greater than C, then A has cardinality greater than C.

(b) Find a sequence of infinite sets $\{A_n\}_{n=1}^{\infty}$ such that for each n, A_{n+1} has cardinality greater than A_n.

(c) Find a set A with cardinality greater than A_n for each of the sets in (b), above.