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Instructions: Answer all of the questions in Part I and two of the questions in Part II. The questions in Part I are worth 10 points each. The questions in Part II are worth 15 points each. There are 110 total points available.

Part I

Answer all of the following questions.

1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. Suppose there is an infinite countable subset \( S \subseteq \mathbb{R} \) such that
   \[
   \int_a^b f(x) \, dx = 0
   \]
   whenever \( a, b \notin S \). Show that \( f = 0 \).

2. Let \( C([-1,1]) \) denote the Banach space of continuous real-valued functions on \([-1,1]\) equipped with the supremum norm. Determine whether each of the following sets is dense in \( C([-1,1]) \) and justify your answer:
   (a) \( \text{span}\{1, x^2, x^4, x^6, \ldots\} \)
   (b) \( \text{span}\{1, x^{171}, x^{172}, x^{173}, \ldots\} \)

3. Give an example of a sequence \( (f_n)_{n=1}^\infty \) of non-negative measurable functions on \( \mathbb{R} \) and a measurable function \( f \) on \( \mathbb{R} \) such that
   
   i. \( f_{n+1}(x) \leq f_n(x) \) for all \( n \geq 1 \) and \( x \in \mathbb{R} \), and
   
   ii. \( \lim_{n \to \infty} f_n(x) = f(x) \) for all \( x \in \mathbb{R} \),

   but
   \[
   \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) \, dx \neq \int_{\mathbb{R}} f(x) \, dx.
   \]

4. Let \( (X, d) \) be a complete countable metric space. Show there is \( x \in X \) such that the singleton \( \{x\} \) is open.

5. Let \( X \) and \( Y \) be topological spaces such that \( X \) is compact and \( Y \) is Hausdorff. Let \( f : X \to Y \) be a continuous bijection. Show that \( f \) is a homeomorphism.

6. Evaluate
   \[
   \int_{0}^{2\pi} \frac{1}{1 + \cos \theta} \, d\theta
   \]
7. Let $X$ and $Y$ be non-empty sets. Let

$$\mathcal{F} = \{(A, B, f) \mid A \subseteq X, B \subseteq Y, f : A \to B \text{ is a bijection}\}.$$ 

Partially order $\mathcal{F}$ by $(A_1, B_1, f_1) \preceq (A_2, B_2, f_2)$ if and only if $A_1 \subseteq A_2$, $B_1 \subseteq B_2$ and $f_2$ restricts to $f_1$ on $A_1$. Use this to show that one of the following two possibilities must hold:

i. There exists a one-to-one function from $X$ into $Y$.

ii. There exists an onto function from $X$ onto $Y$.

8. (a) Let $(X, d)$ be a metric space and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on $(X, d)$ that converges uniformly to a function $f : X \to \mathbb{R}$. Show that $f$ is also continuous.

(b) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Suppose that this series converges at some $x_0 \in \mathbb{R}$ with $x_0 \neq 0$. Show that the power series converges for every $x \in (-|x_0|, |x_0|)$.

(c) Define $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-|x_0|, |x_0|)$, where the power series and $x_0 \in \mathbb{R}$ are as in (b). Show that $f$ is continuous on $(-|x_0|, |x_0|)$.

Part II

Answer two of the following questions.

1. (a) i. Prove Liouville's theorem that a bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.

ii. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that the range of $f$ is dense in $\mathbb{C}$.

iii. Show that a bounded harmonic function $u : \mathbb{R}^2 \to \mathbb{R}$ is constant.

(b) i. Show that for every non-constant polynomial $p \in \mathbb{C}[z]$,

$$\lim_{|z| \to \infty} |p(z)| = \infty.$$ 

ii. Prove the Fundamental Theorem of Algebra: For every non-constant polynomial $p \in \mathbb{C}[z]$, there is $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

2. Let $m$ denote the Lebesgue measure on $\mathbb{R}$.

(a) Let $E \subseteq \mathbb{R}$ be a measurable set with $0 < m(E) < \infty$. Show that the function

$$F(x) = m((x + E) \cap E)$$ 

is continuous at $x = 0$, where $x + E = \{x + y \mid y \in E\}$.

(b) Let $E \subseteq \mathbb{R}$ be a measurable set with $m(E) > 0$. Show that the set

$$E - E = \{x - y \mid x, y \in E\}$$

contains an open interval $(-\delta, \delta)$ for some $\delta > 0$. 

2
(c) Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function such that \( f(x) + f(y) = f(x + y) \) for all \( x, y \in \mathbb{R} \). Show that \( f \) is continuous.

(d) Let \( f \) be as in (c). Show there is \( \gamma \in \mathbb{R} \) such that \( f(x) = \gamma x \) for every \( x \in \mathbb{R} \).

[15] 3. Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be normed spaces, and let \( T : X \to Y \) be a linear map. Say that \( T \) is \textit{bounded} if the quantity

\[
\|T\| := \sup_{\|x\|_X \leq 1} \|Tx\|_Y < \infty
\]

is finite.

(a) Prove that the following are equivalent:
   i. \( T \) is continuous.
   ii. \( T \) is continuous at 0.
   iii. \( T \) is bounded.

(b) Let \((\mathbb{R}^n, \| \cdot \|_2)\) denote the usual Euclidean space. A matrix \( A \in \mathbb{R}^{n \times n} \) gives rise to a linear map \( A : \mathbb{R}^n \to \mathbb{R}^n \) in the usual way, so that the norm of \( A \) can be defined as above by

\[
\|A\| := \sup_{\|x\|_2 \leq 1} \|Ax\|_2 < \infty.
\]

   i. Let

\[
D = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{bmatrix} \in \mathbb{R}^{n \times n}
\]

be a diagonal matrix. Show that \( \|D\| = \max\{d_1, d_2, \ldots, d_n\} \).

   ii. Let \( D \) be as in (i). Show that

\[
\|D\| = \sup_{\|x\|_2 \leq 1} |(Dx, x)|.
\]

   iii. Let \( U \in \mathbb{R}^{n \times n} \) be an orthogonal matrix, i.e. a matrix satisfying \( U^T U = I \), where \( U^T \) denotes the transpose of \( U \). Show that for every \( x \in \mathbb{R}^n \), \( \|Ux\| = \|x\| \).

   iv. Let \( A \in \mathbb{R}^{n \times n} \) be a matrix and let \( \alpha \) denote the largest eigenvalue of the matrix \( A^T A \). Show that \( \|A\| = \sqrt{|\alpha|} \).

   v. Compute \( \|A\| \) for

\[
A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.
\]