University of Waterloo
Department of Pure Mathematics
Analysis & Topology Comprehensive Exam
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Instructions: Answer all of the questions in Part I and two of the questions in Part II. The questions in Part I are worth 10 points each. The questions in Part II are worth 15 points each. There are 110 total points available.

Part I

Answer **all** of the following questions.

[10] 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose there is an infinite countable subset $S \subseteq \mathbb{R}$ such that

$$\int_a^b f(x)\, dx = 0$$

whenever $a, b \notin S$. Show that $f = 0$.

[10] 2. Let $C([-1,1])$ denote the Banach space of continuous real-valued functions on $[-1,1]$ equipped with the supremum norm. Determine whether each of the following sets is dense in $C([-1,1])$ and justify your answer:

(a) $\text{span}\{1, x^2, x^4, x^6, \ldots\}$

(b) $\text{span}\{1, x^{171}, x^{172}, x^{173}, \ldots\}$

[10] 3. Give an example of a sequence $(f_n)_{n=1}^{\infty}$ of non-negative measurable functions on $\mathbb{R}$ and a measurable function f on $\mathbb{R}$ such that

i. $f_{n+1}(x) \le f_n(x)$ for all $n \ge 1$ and $x \in \mathbb{R}$, and

ii. $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$,

but

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx \neq \int_{\mathbb{R}} f(x)\, dx.$$

[10] 4. Let (X, d) be a complete *countable* metric space. Show there is $x \in X$ such that the singleton $\{x\}$ is open.

[10] 5. Let X and Y be topological spaces such that X is compact and Y is Hausdorff. Let $f : X \to Y$ be a continuous bijection. Show that f is a homeomorphism.

[10] 6. Evaluate

$$\int_0^{2\pi} \frac{1}{1+\cos\theta}\, d\theta.$$

[10] 7. Let X and Y be non-empty sets. Let

$$\mathcal{F} = \{(A, B, f) \mid A \subseteq X,\ B \subseteq Y,\ f : A \to B \text{ is a bijection}\}.$$

Partially order $\mathcal{F}$ by $(A_1, B_1, f_1) \preceq (A_2, B_2, f_2)$ if and only if $A_1 \subseteq A_2$, $B_1 \subseteq B_2$ and f_2 restricts to f_1 on A_1. Use this to show that one of the following two possibilities must hold:

i. There exists a one-to-one function from X into Y.

ii. There exists an onto function from X onto Y.

[10] 8. (a) Let (X, d) be a metric space and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on (X, d) that converges uniformly to a function $f : X \to \mathbb{R}$. Show that f is also continuous.

(b) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Suppose that this series converges at some $x_0 \in \mathbb{R}$ with $x_0 \neq 0$. Show that the power series converges for every $x \in (-|x_0|, |x_0|)$.

(c) Define $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-|x_0|, |x_0|)$, where the power series and $x_0 \in \mathbb{R}$ are as in (b). Show that f is continuous on $(-|x_0|, |x_0|)$.

Part II

Answer two of the following questions.

[15] 1. (a) i. Prove Liouville's theorem that a bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.

ii. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that the range of f is dense in $\mathbb{C}$.

iii. Show that a bounded harmonic function $u : \mathbb{R}^2 \to \mathbb{R}$ is constant.

(b) i. Show that for every non-constant polynomial $p \in \mathbb{C}[z]$,

$$\lim_{|z| \to \infty} |p(z)| = \infty.$$

ii. Prove the Fundamental Theorem of Algebra: For every non-constant polynomial $p \in \mathbb{C}[z]$, there is $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

2. Let m denote the Lebesgue measure on $\mathbb{R}$.

[15] (a) Let $E \subseteq \mathbb{R}$ be a measurable set with $0 < m(E) < \infty$. Show that the function

$$F(x) = m((x + E) \cap E)$$

is continuous at $x = 0$, where $x + E = \{x + y \mid y \in E\}$.

(b) Let $E \subseteq \mathbb{R}$ be a measurable set with $m(E) > 0$. Show that the set

$$E - E = \{x - y \mid x, y \in E\}$$

contains an open interval $(-\delta, \delta)$ for some $\delta > 0$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(x) + f(y) = f(x+y)$ for all $x, y \in \mathbb{R}$. Show that f is continuous.

(d) Let f be as in (c). Show there is $\gamma \in \mathbb{R}$ such that $f(x) = \gamma x$ for every $x \in \mathbb{R}$.

[15] 3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and let $T : X \to Y$ be a linear map. Say that T is *bounded* if the quantity

$$\|T\| := \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|Tx\|_Y < \infty$$

is finite.

(a) Prove that the following are equivalent:

i. T is continuous.

ii. T is continuous at 0.

iii. T is bounded.

(b) Let $(\mathbb{R}^n, \|\cdot\|_2)$ denote the usual Euclidean space. A matrix $A \in \mathbb{R}^{n \times n}$ gives rise to a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ in the usual way, so that the norm of A can be defined as above by

$$\|A\| := \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 \leq 1}} \|Ax\|_2 < \infty.$$

i. Let

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

be a diagonal matrix. Show that $\|D\| = \max\{d_1, d_2, \ldots, d_n\}$.

ii. Let D be as in (i). Show that

$$\|D\| = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 \leq 1}} |\langle Dx, x \rangle|.$$

iii. Let $U \in \mathbb{R}^{n \times n}$ be an orthogonal matrix, i.e. a matrix satisfying $U^T U = I$, where U^T denotes the transpose of U. Show that for every $x \in \mathbb{R}^n$, $\|Ux\| = \|x\|$.

iv. Let $A \in \mathbb{R}^{n \times n}$ be a matrix and let α denote the largest eigenvalue of the matrix $A^T A$. Show that $\|A\| = \sqrt{|\alpha|}$.

v. Compute $\|A\|$ for

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$