Department of Pure Mathematics

Algebra Comprehensive Examination

2:30-5:30pm, January 21, 2015

Prepared by Y.-R. Liu and M. Satriano

Instructions: Answer **seven** of the following eight questions. If you answer all eight, clearly indicate which question you do *not* want marked. In the following, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{Z}$ the set of integers and $\mathbb{N}$ the set of positive integers.

Linear Algebra

1. Let A be a $n \times n$ complex matrix and A^* the *adjoint* of A, i.e., $(A^*)_{ij} = \bar{A}_{ji}$.
   (a) Prove that $I + A^*A$ is invertible, where I is the identity matrix.
   (b) Let $\zeta_n = e^{2\pi i/n}$ be a nth root of 1. Suppose that the ijth entry of A is defined by $A_{ij} = \zeta_n^{ij}/\sqrt{n}$. Prove that A is *unitary*, i.e., $A^*A = I$.

2. Let $T : V \to V$ be a liner transformation of vector spaces. Suppose that for $v \in V$, $T^k(v) = 0$, but $T^{k-1}(v) \neq 0$.
   (a) Prove that the set $S = \{v, T(v), \ldots, T^{k-1}(v)\}$ is linearly independent.
   (b) Prove that the subspace W generated by S is T-invariant.
   (c) Show that the restriction $\widehat{T}$ of T to W is *nilpotent of index k*, i.e., $\widehat{T}^k = 0$ (the zero matrix), but $\widehat{T}^{k-1} \neq 0$. Then write down the matrix of T in the basis $\{T^{k-1}(v), \ldots, T(v), v\}$ of W. Justify your answer.

Group Theory

3. (a) Let G be a finite group, and let p be a prime with $p||G|$. Let n_p be the number of Sylow p-subgroups of G. Show that if $n_p \neq 1$ and $|G|$ does not divide $n_p!$, then G is not simple.
   (b) Prove there are no simple groups of order 80.

4. The following questions explore properties of $\mathbb{Q}$ viewed as a group under addition.
   (a) Prove that $\mathbb{Q}$ (under addition) is not a direct product of any two non-trivial subgroups.
   (b) Let P be the set of primes. Given $\varnothing \neq S \subseteq P$, let G_S be the set of rational numbers of the form a/b with $a, b \in \mathbb{Z}$ relatively prime, $b \neq 0$, and either $b = 1$ or every prime divisor of b is an element of S. Prove that G_S is a subgroup of $\mathbb{Q}$ under addition.
   (c) Show that if S and T are non-trivial subsets of P and $G_S = G_T$, then $S = T$. Conclude that $\mathbb{Q}$ is a countable group with uncountably many subgroups.

Ring Theory

5. Let $R = \mathbb{Z}[\sqrt{-5}]$. Let $\psi : R \to R \bigoplus R$ be the R-module map defined by $\psi(1) = (2, 1 + \sqrt{-5})$ and let M be the cokernel of ψ, i.e., $M \simeq (R \bigoplus R)/\operatorname{im}\psi$.

   (a) Let $\langle 2, 1 + \sqrt{-5}\rangle$ be the ideal of R generated by 2 and $(1 + \sqrt{-5})$. Prove that $\langle 2, 1 + \sqrt{-5}\rangle \neq R$.

   (b) Prove that M does not contain a free sub-module of rank 2.

   (c) Is M a free R-module? Justify your answer with proof.

6. Let $V = \bigoplus_{i\in\mathbb{N}} k$ be a countably infinite dimensional vector space over a field k and let $R = \operatorname{End}_k(V)$.

   (a) Let m be a positive integer and let $f \in R$ be given by $f(a_1, a_2, \dots) = (a_m, a_{m+1}, \dots)$. Prove that the two-sided ideal $\mathcal{J}$ generated by f is R.

   (b) Prove that $\mathcal{K} = \{f \in R \mid \operatorname{rank}(f) < \infty\}$ is a non-trivial two-sided ideal of R.

   (c) Show that if $\mathcal{J}$ is any two-sided ideal of R not contained in $\mathcal{K}$, then $\mathcal{J} = R$.

Fields and Galois Theory

7. Let $n \in \mathbb{N}$ and $f(x) = x^n - p$ with p a prime.

   (a) Find the splitting field E of f over $\mathbb{Q}$. Justify your answer.

   (b) If n is a prime, prove that $[E : \mathbb{Q}] = n(n-1)$.

8. (a) The polynomial $f(x) = x^4 + 2x + 2 \in \mathbb{Q}[x]$ is irreducible. Let E_f be the splitting field of $f(x)$ over $\mathbb{Q}$. Compute the Galois group $\operatorname{Gal}_{\mathbb{Q}}(E_f)$. Justify your answer.

   (b) The polynomial $g(x) = x^4 - 2 \in \mathbb{Q}[x]$ is irreducible. Let E_g be the splitting field of $g(x)$ over $\mathbb{Q}$. Compute the Galois group $\operatorname{Gal}_{\mathbb{Q}}(E_g)$. Justify your answer.