Instructions: Answer seven of the following eight questions. If you answer all eight, clearly indicate which question you do not want marked. In the following, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{Z}$ the set of integers and $\mathbb{N}$ the set of positive integers.

Linear Algebra

1. Let $A$ be a $n \times n$ complex matrix and $A^*$ the adjoint of $A$, i.e., $(A^*)_{ij} = \bar{A}_{ji}$.
   
   (a) Prove that $I + A^*A$ is invertible, where $I$ is the identity matrix.
   
   (b) Let $\zeta_n = e^{2\pi i/n}$ be a $n$th root of 1. Suppose that the $ij$th entry of $A$ is defined by $A_{ij} = \zeta_{ij} n / \sqrt{n}$. Prove that $A$ is unitary, i.e., $A^*A = I$.

2. Let $T : V \to V$ be a linear transformation of vector spaces. Suppose that for $v \in V$, $T^k(v) = 0$, but $T^{k-1}(v) \neq 0$.
   
   (a) Prove that the set $S = \{v, T(v), \ldots, T^{k-1}(v)\}$ is linearly independent.
   
   (b) Prove that the subspace $W$ generated by $S$ is $T$-invariant.
   
   (c) Show that the restriction $\hat{T}$ of $T$ to $W$ is nilpotent of index $k$, i.e., $\hat{T}^k = 0$ (the zero matrix), but $\hat{T}^{k-1} \neq 0$. Then write down the matrix of $T$ in the basis $\{T^{k-1}(v), \ldots, T(v), v\}$ of $W$. Justify your answer.

Group Theory

3. (a) Let $G$ be a finite group, and let $p$ be a prime with $p|||G|$. Let $n_p$ be the number of Sylow $p$-subgroups of $G$. Show that if $n_p \neq 1$ and $|G|$ does not divide $n_p!$, then $G$ is not simple.
   
   (b) Prove there are no simple groups of order 80.

4. The following questions explore properties of $\mathbb{Q}$ viewed as a group under addition.
   
   (a) Prove that $\mathbb{Q}$ (under addition) is not a direct product of any two non-trivial subgroups.
   
   (b) Let $P$ be the set of primes. Given $\emptyset \neq S \subseteq P$, let $G_S$ be the set of rational numbers of the form $a/b$ with $a, b \in \mathbb{Z}$ relatively prime, $b \neq 0$, and either $b = 1$ or every prime divisor of $b$ is an element of $S$. Prove that $G_S$ is a subgroup of $\mathbb{Q}$ under addition.
   
   (c) Show that if $S$ and $T$ are non-trivial subsets of $P$ and $G_S = G_T$, then $S = T$. Conclude that $\mathbb{Q}$ is a countable group with uncountably many subgroups.
Ring Theory

5. Let \( R = \mathbb{Z}[\sqrt{-5}] \). Let \( \psi : R \to R \oplus R \) be the \( R \)-module map defined by \( \psi(1) = (2, 1 + \sqrt{-5}) \) and let \( M \) be the cokernel of \( \psi \), i.e., \( M \simeq (R \oplus R)/\text{im } \psi \).

(a) Let \( \langle 2, 1 + \sqrt{-5} \rangle \) be the ideal of \( R \) generated by 2 and \( 1 + \sqrt{-5} \). Prove that \( \langle 2, 1 + \sqrt{-5} \rangle \neq R \).

(b) Prove that \( M \) does not contain a free sub-module of rank 2.

(c) Is \( M \) a free \( R \)-module? Justify your answer with proof.

6. Let \( V = \bigoplus_{i \in \mathbb{N}} k \) be a countably infinite dimensional vector space over a field \( k \) and let \( R = \text{End}_k(V) \).

(a) Let \( m \) be a positive integer and let \( f \in R \) be given by \( f(a_1, a_2, \ldots) = (a_m, a_{m+1}, \ldots) \). Prove that the two-sided ideal \( J \) generated by \( f \) is \( R \).

(b) Prove that \( \mathcal{K} = \{ f \in R \mid \text{rank}(f) < \infty \} \) is a non-trivial two-sided ideal of \( R \).

(c) Show that if \( J \) is any two-sided ideal of \( R \) not contained in \( \mathcal{K} \), then \( J = R \).

Fields and Galois Theory

7. Let \( n \in \mathbb{N} \) and \( f(x) = x^n - p \) with \( p \) a prime.

(a) Find the splitting field \( E \) of \( f \) over \( \mathbb{Q} \). Justify your answer.

(b) If \( n \) is a prime, prove that \( [E : \mathbb{Q}] = n(n - 1) \).

8. (a) The polynomial \( f(x) = x^4 + 2x + 2 \in \mathbb{Q}[x] \) is irreducible. Let \( E_f \) be the splitting field of \( f(x) \) over \( \mathbb{Q} \). Compute the Galois group \( \text{Gal}_\mathbb{Q}(E_f) \). Justify your answer.

(b) The polynomial \( g(x) = x^4 - 2 \in \mathbb{Q}[x] \) is irreducible. Let \( E_g \) be the splitting field of \( g(x) \) over \( \mathbb{Q} \). Compute the Galois group \( \text{Gal}_\mathbb{Q}(E_g) \). Justify your answer.