ON THE abc CONJECTURE, II

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1. Introduction

Let x, y and z be positive integers and define $G = G(x, y, z)$ by

$$G = G(x, y, z) = \prod_{\substack{p|xyz \\ p \text{ a prime}}} p\ .$$

Thus G is the greatest square-free factor of xyz. In 1985 Masser [6] proposed a refinement of a conjecture which had been recently formulated by Oesterlé. Masser conjectured that for each positive real number ε there is a positive number $c(\varepsilon)$, which depends on ε only, such that for all positive integers x, y and z with

$$x + y = z \quad \text{and} \quad (x, y, z) = 1, \tag{1}$$

we have

$$z < c(\varepsilon)G^{1+\varepsilon}. \tag{2}$$

The conjecture is now known as the *abc* conjecture. It captures in a succinct way the idea that the additive and the multiplicative structure of the integers should be independent and, accordingly, it has profound consequences (cf.[1], [3], [4], [5], [11], [13]).

In 1986, Stewart and Tijdeman [11] obtained an upper bound for z as a function of G. They proved that there exists an effectively computable positive constant c_1 such that for all positive integers x, y and z satisfying (1),

$$z < \exp\left(c_1 G^{15}\right). \tag{3}$$

The proof depends on a p-adic estimate for linear forms in the logarithms of algebraic numbers due to van der Poorten [8]. In 1991, Stewart and Yu [12] strengthened (3). They proved, by combining a p-adic estimate for linear forms in the logarithms of algebraic numbers due to Yu [15] with an earlier Archimedean estimate due to Waldschmidt [14], that there exists an effectively computable positive constant c_2 such that, for all positive integers x, y and z, with $z > 2$, satisfying (1),

$$z < \exp\left(G^{2/3+c_2/\log\log G}\right). \tag{4}$$

Our purpose in this paper is to present two further improvements on (4).

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Theorem 1. *There exists an effectively computable positive number c such that, for all positive integers x, y and z with $x+y=z$ and $(x,y,z)=1$,*

$$z < \exp\left(c\, G^{1/3}(\log G)^3\right). \tag{5}$$

The key new ingredient in our proof of Theorem 1 is an estimate of Yu [17] for p-adic linear forms in the logarithms of algebraic numbers which has a better dependence on the number of terms in the linear form than previous p-adic estimates; for the Archimedean case see the earlier work of Matveev [7]. We employ this estimate in order to control the p-adic order of x, y and z at the small primes p dividing x, y and z. Next we combine together the contributions from the small primes in order to reduce the number of terms in our linear forms. We conclude with a further application of estimates for linear forms in the logarithms of algebraic numbers in a fashion similar to [12]. Here we appeal to a p-adic estimate due to Yu [16] and its earlier Archimedean counterpart due to Baker and Wüstholz [2].

An examination of our proof reveals that the impediment to a further refinement of Theorem 1 is not the dependence on the number of terms in the estimates for linear forms in logarithms but instead is the dependence on the parameter p in the p-adic estimates. This fact is highlighted by our next result which shows that if the greatest prime factor of one of x, y and z is small relative to G then the estimate for z from Theorem 1 can be improved. In particular, let p_x, p_y and p_z denote the greatest prime factors of x, y and z respectively with the convention that the greatest prime factor of 1 is 1. Put

$$p' = \min\{p_x, p_y, p_z\}.$$

Denote the i-th iterate of the logarithmic function by $\log_i$ so that $\log_1 t = \log t$ and $\log_i t = \log(\log_{i-1} t)$ for $i = 2, 3, \dots$.

Theorem 2. *There exists an effectively computable positive number c such that, for all positive integers x, y and z with $x+y=z$, $(x,y,z)=1$ and $z>2$,*

$$z < \exp\left(p' G^{\,c\log_3 G^*/\log_2 G}\right), \tag{6}$$

where $G^* = \max(G, 16)$.

Thus, for each $\varepsilon > 0$ there exists a number $c_3(\varepsilon)$, which is effectively computable in terms of ε, such that, for all positive integers x, y and z with $x+y=z$ and $(x,y,z)=1$,

$$z < \exp\left(c_3(\varepsilon) p' G^{\varepsilon}\right).$$

Observe that

$$p' \le (p_x p_y p_z)^{1/3} \le G^{1/3},$$

and so we immediately obtain

$$z < \exp\left(c_3(\varepsilon) G^{1/3+\varepsilon}\right),$$

a slightly weaker version of Theorem 1. On the other hand, if p' is appreciably smaller than $G^{1/3}$, (6) gives a sharper upper bound than (5).

For any integer n with $n > 1$, let $P(n)$ denote the greatest prime factor of n. As an illustration of the above remark we shall deduce from Theorem 2 that there exists an effectively computable positive number c_4 such that if x and y are coprime positive integers with $x < y$ and $y \ge 16$ then

$$P = P(xy(x+y)) > c_4 \log_2 y \log_3 y / \log_4^* y, \tag{7}$$

where $\log_4^* y = \max\{\log_4 y, 1\}$, a result which improves upon the lower bound of $c_5 \log_2 y$ obtained by Shorey, van der Poorten, Tijdeman and Schinzel [10]. Suppose $y \geq 16$ and (1) holds. Then by (6) there exists an effectively computable positive number c_6 such that

$$\log\log y < \log P + c_6 \log G \log_3 G^* / \log_2 G, \tag{8}$$

where G denotes the greatest square-free factor of $xy(x+y)$. Plainly

$$G = \prod_{p \mid xy(x+y)} p \leq \exp\left(\sum_{p \leq P} \log p\right)$$

and so, by the prime number theorem, we have

$$\log G < c_7 P, \tag{9}$$

where c_7 is an effectively computable positive number; (cf. Theorem 9 of Rosser and Schoenfeld [9]). Estimate (7) follows directly from (8) and (9).

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2. Preliminary lemmas

For any algebraic number α let $h_0(\alpha)$ denote its absolute logarithmic Weil height, so that

$$h_0(\alpha) = d^{-1}\left(\log|a_d| + \sum_{j=1}^{d} \log\max(1, |\alpha^{(j)}|)\right),$$

where the minimal polynomial for α over $\mathbb{Z}$ is

$$a_d x^d + \cdots + a_1 x + a_0 = a_d(x - \alpha^{(1)}) \cdots (x - \alpha^{(d)}).$$

Let p be a prime number and put

$$q = \begin{cases} 2 & \text{if } p > 2 \\ 3 & \text{if } p = 2 \end{cases} \quad \text{and} \quad \alpha_0 = \begin{cases} -1 & \text{if } p \equiv 3 \pmod 4 \\ i = \sqrt{-1} & \text{if } p \equiv 1 \pmod 4 \\ e^{2\pi i/3} & \text{if } p = 2. \end{cases}$$

Put $K = \mathbb{Q}(\alpha_0)$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$, the ring of algebraic integers in K. Suppose that $\mathfrak{p}$ lies above p with ramification index $e_\mathfrak{p}$ and residue class degree $f_\mathfrak{p}$. Note that

$$e_\mathfrak{p} = 1 \quad \text{and} \quad f_\mathfrak{p} = \begin{cases} 1 & \text{if } p > 2 \\ 2 & \text{if } p = 2, \end{cases} \tag{10}$$

see the Appendix of Yu [15] for the case $p \equiv 1 \pmod 4$ and the case $p = 2$. For non-zero α in K we write $\mathrm{ord}_\mathfrak{p}\alpha$ for the exponent to which $\mathfrak{p}$ divides the fractional ideal generated by α in K.

Let $\alpha_1, \ldots, \alpha_n$ be non-zero elements of K and put

$$h_j = \max\big(h_0(\alpha_j), \log p\big),$$

for $j = 1, \ldots, n$. Let $b_1, \ldots, b_n$ be rational integers with absolute values at most $B(\geq 3)$. Next put

$$\Theta = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1.$$

Lemma 1. *Suppose that* $\left[K\left(\alpha_0^{1/q},\dots,\alpha_n^{1/q}\right):K\right]=q^{n+1}$, $\operatorname{ord}_{\mathfrak{p}}\alpha_j=0$ *for* $j=1,\dots,n$ *and* $\Theta\neq 0$. *Then there exists an effectively computable positive number* c_8 *such that*

$$\operatorname{ord}_{\mathfrak{p}}\Theta<\frac{p}{(\log p)^2}\,c_8^n\,h_1\cdots h_n\log B.$$

Proof. This follows from Theorem 1 of Yu [17] on applying (10).

Lemma 2. *Suppose that* $\operatorname{ord}_{\mathfrak{p}}\alpha_j=0$ *for* $j=1,\dots,n$ *and* $\Theta\neq 0$. *Then there exists an effectively computable positive number* c_9 *such that*

$$\operatorname{ord}_{\mathfrak{p}}\Theta<\frac{p}{\log p}(c_9 n)^{2n}\left(\frac{h_1}{\log p}\right)\cdots\left(\frac{h_n}{\log p}\right)\log B.$$

Proof. This follows from Theorem 1 of Yu [16] on appealing to (10).

We shall also require an Archimedean estimate for linear forms in the logarithms of algebraic numbers.

Lemma 3. *Suppose that* $\alpha_1,\dots,\alpha_n$ *are positive rational numbers and put*

$$\Lambda=b_1\log\alpha_1+\cdots+b_n\log\alpha_n,$$

where log *denotes the principal branch of the logarithm. If* $\Lambda\neq 0$ *then there exists an effectively computable positive number* c_{10} *such that*

$$|\Lambda|>\exp\left(-(c_{10}n)^{2n}\log B\prod_{j=1}^{n}\max\left(h_0(\alpha_j),1\right)\right).$$

Proof. This is a consequence of the Theorem of Baker and Wüstholz [2].

Lemma 4. *Let* $\alpha_1,\dots,\alpha_n$ *be prime numbers with* $\alpha_1<\alpha_2<\cdots<\alpha_n$. *Let* $q=2$ *and* $\alpha_0\in\{-1,i\}$ *or* $q=3$ *and* $\alpha_0=e^{2\pi i/3}$ *and put* $K=\mathbb{Q}(\alpha_0)$. *Then*

$$\left[K\left(\alpha_0^{1/q},\alpha_1^{1/q},\dots,\alpha_n^{1/q}\right):K\right]=q^{n+1},$$

except when $q=2$, $\alpha_0=i$ *and* $\alpha_1=2$, *and in this case*

$$\left[K\left(\alpha_0^{1/2},(1+i)^{1/2},\alpha_2^{1/2},\dots,\alpha_n^{1/2}\right):K\right]=2^{n+1}.$$

Proof. This follows from Lemma 3 of [12] except when $q=2$ and $\alpha_0=-1$. In this case the proof of Lemma 3 of [12] again applies.

Lemma 5. *Let* $2=p_1,p_2,\dots$ *be the sequence of prime numbers in increasing order. There is an effectively computable positive constant* c_{11} *such that for every positive integer* r *we have*

$$\prod_{j=1}^{r}\frac{p_j}{\log p_j}>\left(\frac{r+3}{c_{11}}\right)^{r+3}.$$

Proof. This is Lemma 4 of [12].

3. Proof of Theorems 1 and 2

Note that Theorem 1 holds for $z = 2$ trivially. Henceforth let x, y and z be positive integers with $x + y = z$, $(x, y, z) = 1$ and $z > 2$. We may suppose, without loss of generality, that $x \le y$. Since $z > 2$ we see that $x < y < z$ and $G \ge 6$. Note that

$$\max\{\mathrm{ord}_p x, \mathrm{ord}_p y, \mathrm{ord}_p z\} \le \frac{\log z}{\log 2} \ . \tag{11}$$

Put

$$\tilde{G} = \max\left(G/(p_x p_y p_z), 16\right)$$

and

$$r = \omega(xyz),$$

the number of distinct prime factors of xyz.

Let $c_{12}, c_{13}, \ldots$ denote effectively computable positive constants. By Lemma 5

$$\tilde{G} > \left(\frac{r}{c_{12}}\right)^r,$$

and so

$$r < c_{13} \log \tilde{G} / \log_2 \tilde{G}. \tag{12}$$

Put $m = r - 2$ if $x = 1$ (whence $p_x = 1$) and $m = r - 3$ otherwise. Notice that, by the arithmetic-geometric mean inequality,

$$\prod_{\substack{p|xyz \\ p \notin \{p_x, p_y, p_z\}}} \log p \le \left(\frac{1}{m} \sum_{\substack{p|xyz \\ p \notin \{p_x, p_y, p_z\}}} \log p\right)^m \le \left(\frac{\log \tilde{G}}{m}\right)^m, \tag{13}$$

provided that m is positive. From (12) and (13) we deduce that

$$\prod_{\substack{p|xyz \\ p \notin \{p_x, p_y, p_z\}}} \log p < \exp\left(c_{14} \frac{\log \tilde{G} \log_3 \tilde{G}}{\log_2 \tilde{G}}\right), \tag{14}$$

with the usual convention that the empty product is 1. It also follows from (12) that

$$(\log r)^{2r} < \exp\left(c_{15} \frac{\log \tilde{G} \log_3 \tilde{G}}{\log_2 \tilde{G}}\right). \tag{15}$$

We shall now estimate $\mathrm{ord}_p(xyz)$ for each prime p which divides xyz and satisfies

$$p < e^{(\log r)^2}. \tag{16}$$

First suppose that $p|z$. Since $(x, y, z) = 1$ and $x + y = z$ we have $(x, y) = (x, z) = (y, z) = 1$. Thus, for each prime p which divides z,

$$\mathrm{ord}_p z = \mathrm{ord}_p\left(\frac{z}{-y}\right) = \mathrm{ord}_p\left(\frac{x}{-y} - 1\right) \le \mathrm{ord}_p\left(\left(\frac{x}{y}\right)^4 - 1\right). \tag{17}$$

Let $\alpha_1 < \cdots < \alpha_n$ be the primes which divide either x or y except in the case when $p \equiv 1 \pmod 4$ and $\alpha_1 = 2$. In that case we take $\alpha_1 = 1+i$ in place of $\alpha_1 = 2$. Note that $2^4 = (1+i)^8$. Write

$$\left(\frac{x}{y}\right)^4 = \alpha_1^{b_1} \cdots \alpha_n^{b_n} \ ,$$

with $b_1, \dots, b_n$ rational integers. We choose q, α_0, $K = \mathbb{Q}(\alpha_0)$ as in §2. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying above p. Since $p|z$ and $(x,z) = (y,z) = 1$ we have

$$\mathrm{ord}_{\mathfrak{p}} \alpha_j = 0,$$

for $j = 1, \dots, n$. Let B denote the maximum of the absolute values of the b_j's. Then, by (11),

$$\log B \le \log\big(8 \log z / \log 2\big). \tag{18}$$

Put

$$\Theta = \left(\frac{x}{y}\right)^4 - 1 = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1,$$

and note that

$$\mathrm{ord}_p \Theta = \mathrm{ord}_{\mathfrak{p}} \Theta. \tag{19}$$

By (16), (18), Lemma 1 and Lemma 4,

$$\mathrm{ord}_{\mathfrak{p}} \Theta < p\, c_{16}^n (\log r)^{2n} \log\log z \prod_{\substack{l|xy \\ l \text{ a prime}}} \log l. \tag{20}$$

It follows from (12), (14)-(17), (19) and (20), that

$$\mathrm{ord}_p z < \exp\left(c_{17} \frac{\log \tilde{G} \log_3 \tilde{G}}{\log_2 \tilde{G}}\right) \log(2p_x) \log p_y \log\log z. \tag{21}$$

In a similar fashion we deduce, for p satisfying (16), that

$$\mathrm{ord}_p y < \exp\left(c_{18} \frac{\log \tilde{G} \log_3 \tilde{G}}{\log_2 \tilde{G}}\right) \log(2p_x) \log p_z \log\log z, \tag{22}$$

and that

$$\mathrm{ord}_p x < \exp\left(c_{19} \frac{\log \tilde{G} \log_3 \tilde{G}}{\log_2 \tilde{G}}\right) \log p_y \log p_z \log\log z. \tag{23}$$

We now define R, S and T by

$$R = \prod_{\substack{l|x,\, l \ne p_x \\ l < e^{(\log r)^2}}} l^{\,\mathrm{ord}_l x} \ , \quad S = \prod_{\substack{l|y,\, l \ne p_y \\ l < e^{(\log r)^2}}} l^{\,\mathrm{ord}_l y} \ , \quad T = \prod_{\substack{l|z,\, l \ne p_z \\ l < e^{(\log r)^2}}} l^{\,\mathrm{ord}_l z} \ ,$$

where l runs through primes. Observe that

$$h_0\left(\frac{R}{-S}\right) < r(\log r)^2 \max\left(\max_{\substack{l|x \\ l < e^{(\log r)^2}}} \mathrm{ord}_l x \ , \ \max_{\substack{l|y \\ l < e^{(\log r)^2}}} \mathrm{ord}_l y \right),$$

hence, by (12), (22) and (23),

$$h_0\left(\frac{R}{-S}\right) < \exp\left(c_{20}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\log\max(p_x,p_y)\log p_z\log\log z. \tag{24}$$

Similarly, we find that

$$h_0\left(\frac{T}{R}\right) < \exp\left(c_{21}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\log\max(p_x,p_z)\log p_y\log\log z. \tag{25}$$

and that

$$h_0\left(\frac{T}{S}\right) < \exp\left(c_{22}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\log\max(p_y,p_z)\log(2p_x)\log\log z. \tag{26}$$

We are now in a position to estimate $\mathrm{ord}_p(xyz)$ for each prime p which divides xyz. In particular, we no longer require condition (16). We first estimate $\mathrm{ord}_p z$ for $p|z$. As in (17) we have

$$\mathrm{ord}_p z = \mathrm{ord}_p\left(\frac{x}{-y}-1\right). \tag{27}$$

Put $\alpha_1 = R/(-S)$ and let

$$\frac{x}{-y} = \alpha_1\alpha_2^{b_2}\cdots\alpha_n^{b_n},$$

where $\alpha_2,\ldots,\alpha_n$ are distinct prime numbers and $b_2,\ldots,b_n$ are non-zero rational integers. Since

$$\alpha_2\cdots\alpha_n|G$$

and

$$\alpha_j \geq e^{(\log r)^2}\ ,$$

for $j=2,\ldots,n$ with $\alpha_j\notin\{p_x,p_y\}$, we deduce that

$$n-3 \leq \frac{\log\tilde{G}}{(\log r)^2}\ . \tag{28}$$

Next observe that

$$n^{2n} < \exp\left(c_{23}\frac{\log\tilde{G}}{\log_2\tilde{G}}\right)\ , \tag{29}$$

since if r is at most $(\log\tilde{G})^{1/2}$ the result is immediate on noting that n is at most r, while if r exceeds $(\log\tilde{G})^{1/2}$ then (29) follows from (28).

Let $B=\max(|b_2|,\ldots,|b_n|,3)$ and note that (18) follows from (11) as before. Put

$$\Theta = \frac{x}{-y}-1 = \alpha_1\alpha_2^{b_2}\cdots\alpha_n^{b_n}-1,$$

and observe that (19) holds. Next put

$$W_p = \exp\left(c_{24}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\frac{p}{(\log p)^3}\prod_{l\in\{p_x,p_y,p_z\}}\log\max(l,p)\cdot(\log\log z)^2. \tag{30}$$

We now apply Lemma 2, taking into account (12), (14), (24), (27) and (29) to conclude that

$$(\mathrm{ord}_p z)\log p < W_p\log\max(p_x,p_y). \tag{31}$$

Similarly, if $p|y$ then, by considering $\mathrm{ord}_p\bigl(z/x-1\bigr)$ and applying Lemma 2, we find that

$$(\mathrm{ord}_p y)\log p < W_p \log\max(p_x, p_z), \tag{32}$$

while if $p|x$ then, by considering $\mathrm{ord}_p\bigl(z/y-1\bigr)$ and applying Lemma 2, we obtain

$$(\mathrm{ord}_p x)\log p < W_p \log\max(p_y, p_z). \tag{33}$$

Certainly

$$\log z = \sum_{p|z}\left(\mathrm{ord}_p z\right)\log p \le r\left(\max_{p|z}\left(\mathrm{ord}_p z\right)\log p\right). \tag{34}$$

Put

$$L = \log\max(p_x, p_y)\cdot\log\max(p_x, p_z)\cdot\log\max(p_y, p_z).$$

By (12), (30), (31) and (34) we find that

$$\frac{\log z}{(\log\log z)^2} \quad < \quad \exp\left(c_{25}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\frac{p_z}{(\log p_z)^2}L. \tag{35}$$

Since $y > z/2$ and $z \ge 3$,

$$\log y > \log z - \log 2 > \frac{1}{4}\log z. \tag{36}$$

Plainly (34) holds with z replaced by y and so from (12), (30), (32) and (36) we deduce that

$$\frac{\log z}{(\log\log z)^2} \quad < \quad \exp\left(c_{26}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\frac{p_y}{(\log p_y)^2}L. \tag{37}$$

Next, either $x \ge y^{1/2}$ in which case

$$\log x \ge \frac{1}{2}\log y > \frac{1}{8}\log z, \tag{38}$$

or $x < y^{1/2}$ in which case

$$\log\left(\frac{x+y}{y}\right) = \log\left(1+\frac{x}{y}\right) < \log\left(1+\frac{1}{y^{1/2}}\right) < \frac{1}{y^{1/2}} < \left(\frac{2}{z}\right)^{1/2}. \tag{39}$$

In the former case we may appeal to (34) with z replaced by x and so from (12), (30), (33) and (38)

$$\frac{\log z}{(\log\log z)^2} \quad < \quad \exp\left(c_{27}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\frac{p_x}{\bigl(\log(2p_x)\bigr)^2}L. \tag{40}$$

In the latter case, put $\alpha_1 = T/S$ and write

$$\frac{z}{y} = \alpha_1\alpha_2^{b_2}\cdots\alpha_n^{b_n},$$

where $\alpha_2, \ldots, \alpha_n$ are distinct prime numbers and $b_2, \ldots, b_n$ are non-zero rational integers. Then

$$0 < \log\left(\frac{x+y}{y}\right) = \log\left(\frac{z}{y}\right) = \log\alpha_1 + b_2\log\alpha_2 + \cdots + b_n\log\alpha_n.$$

Note that we again have (29). Thus on applying Lemma 3 and appealing to (11), (12), (14), (26) and (29) we obtain

$$\log\log\left(\frac{x+y}{y}\right) > -\exp\left(c_{28}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\log\max(p_y,p_z)\cdot \cdot\log(2p_x)\log p_y\log p_z(\log\log z)^2. \qquad (41)$$

On comparing (39) and (41) we see that

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{29}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\log\max(p_y,p_z)\log(2p_x)\log p_y\log p_z.$$

Therefore, in both cases $x \geq y^{1/2}$ and $x < y^{1/2}$, (40) holds.

Suppose that $\{p_x, p_y, p_z\} = \{p', p'', p'''\}$ and that

$$p' < p'' < p'''.$$

It follows from (35), (37) and (40) that

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{30}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\frac{p'}{\big(\log(2p')\big)^2}\log p''(\log p''')^2. \qquad (42)$$

Just as for (14), we have

$$\prod_{p|xyz}\log p < \exp\left(c_{31}\frac{\log G\log_3 G^*}{\log_2 G}\right),$$

whence, by (42),

$$\frac{\log z}{(\log\log z)^2} < \exp\left(c_{32}\frac{\log G\log_3 G^*}{\log_2 G}\right)\frac{p'}{\big(\log(2p')\big)^2}. \qquad (43)$$

Theorem 2 follows directly from (43).

To prove Theorem 1 we remark that from (35), (37) and (40),

$$\left(\frac{\log z}{(\log\log z)^2}\right)^3 < \exp\left(c_{33}\frac{\log\tilde{G}\log_3\tilde{G}}{\log_2\tilde{G}}\right)\frac{p_xp_yp_z}{\big(\log(2p_x)\log p_y\log p_z\big)^2}(\log p'')^3(\log p''')^6.$$

Note that we may assume that

$$p' > G^{1/4},$$

since otherwise Theorem 1 follows from (43). Thus we have

$$\left(\frac{\log z}{(\log\log z)^2}\right)^3 < c_{34}\,\tilde{G}\,p_xp_yp_z(\log G)^3$$

and so

$$\log z < c_{35}\,G^{1/3}(\log G)^3,$$

as required.

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