The Fourier Algebra of a Locally Compact Group

John Sawatzky

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Abstract

Given a locally compact group $G$, the Fourier algebra $A(G)$ is a closed ideal inside the Fourier-Stieltjes algebra $B(G)$ consisting of the matrix coefficients associated with the left regular representation. The Fourier algebra possesses a number of interesting properties, such as the fact that its spectrum is just $G$ itself, that it has a bounded approximate identity if and only if $G$ is amenable, and that if $H$ is a closed subgroup of $G$ then the map of restriction is surjective: $A(G)|_H = A(H)$. 
Contents

1 Introduction 3

2 Preliminaries 4

3 The Fourier Algebra 9
   3.1 The Construction of the Fourier-Stieltjes Algebra . . . . . . . 9
   3.2 The Fourier Algebra . . . . . . . . . . . . . . . . . . . . . . . . . . 11

4 The Spectrum of $A(G)$ 18

5 Leptin’s Theorem 21

6 Restriction Map to Subgroups 27
1 Introduction

Let $G$ be a locally compact group. In this project we will construct the Fourier-Stieltjes algebra $B(G)$ and the Fourier algebra $A(G)$, which are Banach algebras that serve to generalize the notion of Pontryagin duality from the theory of abelian groups. In particular, we will focus on $A(G)$. Our approach will for the most part be the same as that of Kaniuth and Lau [7], which is actually the initial approach by Eymard [4].

The Fourier algebra is a fairly well-behaved algebra; an application of Fell’s Absorption Principle yields that $A(G)$ is actually an ideal inside of $B(G)$. Furthermore, a lemma of Eymard tells us that $A(G)$ is a regular algebra. One reason to think that $A(G)$ is a “good choice” for the generalization of Pontryagin duality is because the Gelfand spectrum of $A(G)$ behaves very nicely. Indeed, we can show that the spectrum can be identified with exactly $G$ itself. Furthermore, we will see that $A(G)$ behaves nicely with subgroups in the expected way. That is, if $H$ is a closed subgroup of $G$ then not only is $A(G)|_H \subseteq A(H)$, but actually $A(G)|_H = A(H)$. Our proof of that result will follow that of Spronk [9], which is itself based on the approach of [2].

Much can be said about $A(G)$ if we restrict our focus to specific classes of groups. If $G$ is compact then $A(G) = B(G)$, and if $G$ is abelian then $A(G) = L^1(\hat{G})$, where $\hat{G}$ is the Pontryagin dual of $G$. The case where $G$ is amenable turns out to be particularly illuminating, because a theorem of Leptin allows us to completely characterize when $A(G)$ possesses a bounded (indeed, contractive) approximate identity.
This document is written with the implicit assumption that the reader is familiar with all fundamental results from functional analysis and abstract measure theory. Knowledge of abstract harmonic analysis would be helpful, however a brief introduction to the topic is done in the preliminaries section. In particular, a newcomer to this subject should take note of the equivalent definitions of amenability that we will be using.

## 2 Preliminaries

**Definition 2.1.** For a given group $G$ and topology $\tau$ on $G$, we call $(G, \tau)$ a *topological group* (usually just denoted $G$ if the topology is clear) if group multiplication and the taking of inverses is continuous with respect to $\tau$.

We will list a collection of important and easy to prove properties of topological groups.

**Proposition 2.2.** Suppose that $G$ is a topological group. Then the following hold.

1. If $U \subseteq G$ is open and $A \subseteq G$ is any set, then the translated sets $AU = \{ay : a \in A, y \in U\}$ and $UA = \{ya : a \in A, y \in U\}$ are open. In particular, translating an open set by a single element results in another open set.

2. If $U$ is an open neighborhood of $e$ then there exists an open neighborhood $V$ of $e$ such that $VV \subseteq U$ and $V$ is symmetric, that is, $V = \{y^{-1} : y \in V\}$.

3. If $C$ and $K$ are compact subsets of $G$, then $KC$ is compact.

4. If $C$ is compact in $G$ and $U$ is an open set such that $C \subseteq U$, then there exists a symmetric neighborhood $V$ of $e$ such that $CV^2 \cup V^2C \subseteq U$. 
Following the example of most abstract harmonic analysts, from here on out we will assume that $G$ is always a locally compact group. The reason for this restriction of focus is because of the following fundamental result.

**Theorem 2.3 (Existence of Haar Measure).** If $G$ is a locally compact group, then there exists a positive Radon measure $m$ that satisfies the condition that if $x \in G$ and $E$ is measurable, then $m(xE) = m(E)$ (this property is called left-invariance). Furthermore, $m$ is unique up to scaling and if $U$ is an open set then $m(U) > 0$.

While technically this theorem gives us an infinite number of choices for our measure because of the option of scaling, the actual explicit choice of a specific measure is typically not important. Thus without ambiguity for a given locally compact group $G$ we can work with the measure $m$ given by the above theorem, which we will call *Haar measure*. Now from elementary measure theory we know that we can in fact integrate against that measure. That is, if $f$ is an integrable function on $G$ then we can write the integral $\int_G f(x) dm(x)$, which we will call the *Haar integral*. Because it is understood that we are always integrating against Haar measure, we will typically omit the actual measure $m$ and just write $\int_G f(x) dx$. Following typical notational convention, if the exact form of $f$ is left arbitrary then we will allow ourselves to write $\int_G f$ instead.

At this point we want to set some notation that will be used throughout this project. Let $f : G \to \mathbb{C}$ and $x \in G$. Then let

\[
\begin{align*}
\check{f}(x) &= f(x^{-1}) \\
\tilde{f}(x) &= \overline{f(x^{-1})} \\
R_x f(y) &= f(yx)
\end{align*}
\]
Remark 1. It turns out that if we consider the above defined ∗ operation (also called convolution) as algebra multiplication, we can actually consider $L^1(G)$ as a Banach algebra.

Note that if $x \in G$ then the map $E \mapsto m(Ex)$ also defines a measure on $G$, hence by uniqueness up to scaling we can find a constant $\Delta(x)$ such that $m(Ex) = \Delta(x)m(E)$. This defines a homomorphism $\Delta : G \to \mathbb{R}$ which we will call the modular function. Due to the following proposition, this map is actually quite useful.

**Proposition 2.4.** If $f \in L^1(G)$ then

$$
\int_G \frac{1}{\Delta(x)} \hat{f}(x)dx = \int_G f(x)dx
$$

With Haar measure in hand, we can construct the Lebesgue spaces $L^p(G, m)$ (which we will simply write as $L^p(G)$ because Haar measure is understood). We recall that in the case of $p = 2$ then $L^2(G)$ is a Hilbert space with inner product $\langle f, g \rangle = \int_G f(x)\overline{g(x)}dx$.

**Proposition 2.5.** The following are equivalent:

i. If $K$ is a compact subset of $G$ and $\epsilon > 0$, then there exists a compact set $C$ such that $m(KC) < (1 + \epsilon)m(C)$;

ii. If $f \in C_c^+(G)$, then $\|f\|_1 = \|\lambda(f)\|$. 

The above equivalent conditions are equivalent to a notion of a group being amenable. There are many, many different equivalent conditions for being amenable which there is not time to get into here, but are covered extensively
in [8].

For a Hilbert space $\mathcal{H}$, let $\mathcal{U}(\mathcal{H}) = \{U \in \mathcal{B}(\mathcal{H}) : U^*U = I = UU^*\}$ be the unitary operators on $\mathcal{H}$. A continuous unitary representation on $G$ is a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H}(\pi))$ such that for any $\zeta, \eta \in \mathcal{H}(\pi)$ then the function $\pi_{\zeta,\eta}(x) = \langle \pi(x)\zeta, \eta \rangle$ is continuous (we often call these functions matrix coefficients). We will denote the collection of equivalence classes of unitary representations on $G$ by $\Sigma(G)$, where two representations are equivalent if they are equal up to a conjugation by a unitary.

Recall that if we have two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we can form the tensor product $\mathcal{H} \otimes \mathcal{K}$ with inner product

$$\langle \zeta \otimes \eta, \phi \otimes \psi \rangle_{\mathcal{H} \otimes \mathcal{K}} = \langle \zeta, \phi \rangle_{\mathcal{H}} \langle \eta, \psi \rangle_{\mathcal{K}}$$

where $\zeta, \phi \in \mathcal{H}$ and $\eta, \psi \in \mathcal{K}$.

Then given two unitary representations $\pi$ and $\rho$, we can form another (continuous unitary) representation called the tensor product of $\pi$ and $\rho$ denoted by $\pi \otimes \rho : G \to \mathcal{U}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$ defined as

$$\pi \otimes \rho(x)(\zeta \otimes \eta) = \pi(x)\zeta \otimes \rho(x)\eta$$

where $\zeta \in \mathcal{H}_\pi, \eta \in \mathcal{H}_\rho, x \in G$.

If we have a continuous unitary representation $\pi$ on $G$, then we can determine a non-degenerate $\ast$–representation on $L^1(G)$ by writing $\pi(f) = \int_G f(x)\pi(x)dx$. This a bit of abuse of notation because the resulting integral is operator-valued, so the method of actually working with this integral in practice is by letting
\( \pi(f) \) be defined by satisfying

\[
\langle \pi(f) \zeta, \eta \rangle = \int_G f(x) \langle \pi(x) \zeta, \eta \rangle dx, \quad \text{where } \zeta, \eta \in \mathcal{H}(\pi) \text{ and } f \in L^1(G)
\]

A unitary representation of particular importance to this project is known as the *left regular representation* and is defined by \( \lambda : G \to U(L^2(G)) \), where if \( x, y \in G \) and \( f \in L^2(G) \) then \( \lambda(y)f(x) = f(y^{-1}x) \). Using the notion above of representations defining operator valued integrals on \( L^1(G) \), we can observe the useful fact that \( \lambda(f)g = f \ast g \), where \( f \in L^1(G) \) and \( g \in L^2(G) \).

Assume that \( G \) is abelian. Let the group

\[
\hat{G} = \{ \sigma : G \to \mathbb{T} : \sigma \text{ a continuous homomorphism} \}
\]

be equipped with the topology of uniform convergence on compact sets. We call \( \hat{G} \) the *dual group of \( G \).* For \( f \in L^1(G) \), let the map \( \hat{f} : \hat{G} \to \mathbb{C} \), called the *Fourier transform of \( f \)*, be defined by \( \hat{f}(\sigma) = \int_G f\sigma dm \).

**Theorem 2.6** (Plancherel’s Theorem). Let \( G \) be abelian and \( f \in L^1(G) \cap L^2(G) \). Then \( \| \hat{f} \|_{L^2(\hat{G})} = \| f \|_{L^2(G)} \). Furthermore, there is a unitary \( U : L^2(G) \to L^2(\hat{G}) \) such that \( Uf = \hat{f} \).

**Theorem 2.7** (Pontryagin duality). If \( G \) is abelian, then \( G \cong \hat{\hat{G}} \).

Before we end this section, we will quickly mention some fundamental theorems from the theory of von Neumann algebras.

**Theorem 2.8** (Kaplansky’s Density Theorem). Let \( \mathcal{H} \) be a Hilbert space. If \( \mathcal{A} \) is a self adjoint algebra of operators in \( \mathcal{B}(\mathcal{H}) \) then \( B_1(\mathcal{A}^{\text{SOT}}) = \overline{B_1(\mathcal{A})}^{\text{SOT}} \),
and in fact, if \( T \in B_1(\overline{A}^{SOT}) \) is self adjoint then it is in the strong operator closure of the self adjoint operators in \( B_1(A) \).

**Theorem 2.9** (von Neumann’s Double Commutant Theorem). Let \( \mathcal{H} \) be a Hilbert space and \( A \subseteq B(\mathcal{H}) \) be a non-degenerate *-subalgebra. Denote the commutant of \( A \) by \( A' = \{ T \in B(\mathcal{H}) : TS = ST \text{ for each } S \in A \} \). Then

\[
\overline{A}^{SOT} = \overline{A}^{WOT} = A''
\]

### 3 The Fourier Algebra

#### 3.1 The Construction of the Fourier-Stieltjes Algebra

Let \( f \in L^1(G) \). Then we define the norm

\[
\| f \|_* = \sup \{ \| \pi(f) \| : \pi \in \Sigma(G) \}
\]

In general \( L^1(G) \) is not complete under this new norm, so we let \( C^*(G) \) denote the completion of \( L^1(G) \) with respect to this \( \| \cdot \|_* \) norm. The space \( C^*(G) \) is called the *group C*-algebra of \( G \). Now define a space of continuous functions

\[
B(G) = \{ \pi_{\zeta, \eta} : \pi \in \Sigma(G) \text{ and } \zeta, \eta \in \mathcal{H}_\pi \}
\]

**Lemma 3.1.** If \( u \in B(G) \), then \( \sup \{ |\int_G f(x)u(x)dx| : f \in L^1(G), \| f \|_* \leq 1 \} < \infty \)

**Proof.** Let \( u(x) = \pi_{\zeta, \eta} \). If \( f \in L^1(G), \| f \|_* \leq 1 \) then we have that

\[
|\int_G f(x)u(x)dx| = |\langle (f)\zeta, \eta \rangle|
\]
3.1 The Construction of the Fourier-Stieltjes Algebra

\[ \leq \|f\|_* \|\zeta\| \|\eta\| \]

\[ \leq \|\zeta\| \|\eta\| \]

This result allows us to equip \( B(G) \) with the following norm. If \( u \in B(G) \) then set

\[ \|u\|_{B(G)} = \sup \{ |\int_G f(x)u(x)dx : f \in L^1(G), \|f\|_* \leq 1 \} \]

We call the normed space \( (B(G), \| \cdot \|_{B(G)}) \) the Fourier-Stieltjes Algebra of \( G \).

To justify the use of the word algebra here, we just note that if \( u = \pi_{\zeta,\eta} \) and \( v = \rho_{\phi,\psi} \) where \( \zeta, \eta \in \mathcal{H}_\pi \) and \( \phi, \psi \in \mathcal{H}_\rho \) then we have that

\[ uv = \langle \pi\zeta, \eta \rangle \langle \rho\phi, \psi \rangle = \langle \pi \otimes \rho(\zeta \otimes \phi), \eta \otimes \psi \rangle \in B(G) \]

For \( \pi \in \Sigma(G) \), define the subspace

\[ A_\pi(G) = \text{span}\{ \pi_{f,g} : f, g \in L^2(G) \}^{B(G)} \]

Let us also define a subalgebra of \( B(\mathcal{H}_\pi) \) by

\[ VN_\pi(G) = \pi(L^1(G))^{\text{WOT}} = \pi(L^1(G))^{\text{w*}} \]

This definition is permissible because the weak* and weak operator topologies agree on bounded sets in \( B(\mathcal{H}_\pi) \). Now with these two spaces in hand, we can look at the following useful theorem:
Theorem 3.2. If \( \pi \in \Sigma(G) \), then

i. \( A_{\pi}(G)^* \cong VN_{\pi}(G) \) via the dual pairing \( \langle \pi_\zeta, \eta \rangle = \langle T \zeta, \eta \rangle \)

ii. \( A_\pi = \{ u \in B(G) : u = \sum_{i=1}^{\infty} \pi_{\zeta_i, \eta_i} \text{ where } \sum_{i=1}^{\infty} \| \zeta_i \| \cdot \| \eta_i \| < \infty \} \)

The proof of this result is outside the scope of this project. The curious reader might enjoy a reference that actually focuses on the Fourier-Stieltjes algebra specifically, such as the one by Anderson-Sackaney [1].

3.2 The Fourier Algebra

While \( B(G) \) is a perfectly interesting algebra in its own right, for the purposes of this project we want to focus on a particular subalgebra of \( B(G) \). Recall that \( \lambda : G \to U(L^2(G)) \), called the left regular representation, is the map such that \( \lambda(x)f(y) = f(x^{-1}y) \) where \( x, y \in G \) and \( f \in L^2(G) \). Using the definitions from the last section, we will let \( A(G) = A_{\lambda}(G) \) be called the Fourier Algebra of \( G \) and \( VN(G) = VN_{\lambda}(G) \) the von Neumann algebra of \( G \).

We will have to justify the use of the word “algebra” here, because if \( \pi \in \Sigma(G) \) then it is by no means necessarily true that \( A_{\pi} \) is any more than just a closed subspace, which gives us an indication of how special a representation \( \lambda \) really is. It turns out that \( \lambda \) possesses a remarkable “absorbing” property that allows \( A(G) \) to not only be a subalgebra of \( B(G) \), but actually an ideal. In order to show this, we will need a theorem by Fell.

Theorem 3.3 (Fell’s Absorption Principle). Let \( \pi \in \Sigma(G) \) and \( \{ e_i \}_{i \in I} \) be an orthonormal basis for \( \mathcal{H}_\pi \). Then we have that \( \pi \otimes \lambda \cong I \cdot \lambda \), where \( I \cdot \lambda : G \to U(\mathcal{H}_\pi)^I \) is the unitary representation defined by \( I \cdot \lambda(x)(\zeta_i)_{i \in I} = (\lambda(x)\zeta_i)_{i \in I} \).
3.2 The Fourier Algebra

Proof. We begin by fixing \( x \in G \) and defining the three following unitaries:

\[
V : L^2(G)^I \to L^2(G) \otimes^2 \mathcal{H}_\pi, \quad V(f_i)_{i \in I} = \sum_{i \in I} f_i \otimes e_i
\]

\[
U : L^2(G) \otimes^2 \mathcal{H}_\pi \to L^2(G, \mathcal{H}_\pi), \quad U\left(\sum_{i \in I} f_i \otimes e_i\right) = \sum_{i \in I} f_i(\cdot) e_i
\]

\[
W : L^2(G, \mathcal{H}_\pi) \to L^2(G, \mathcal{H}_\pi), \quad W(H) = \pi(x) H
\]

It is not hard to see why each of these maps are unitaries. It follows easily that \( V \) is an isometry and has dense range, but because isometries must always have closed range then it follows that \( V \) is a surjective isometry, hence a unitary. As for \( U \), it is easy to see that \( U^{-1} H = \sum_{i \in I} \langle H(\cdot), e_i \rangle \otimes e_i \) and by elementary computation \( U^{-1} = U^* \).

I claim that the unitary \( L = V^* U^* W^* U \) satisfies \( L(\lambda \otimes \pi)L^* = I \cdot \lambda \), which would show that \( \lambda \otimes \pi \) and \( I \cdot \lambda \) are unitarily equivalent. This calculation requires multiple steps, so for notational convenience after each successive application of one of the above unitaries and its adjoint I will denote the result as a function \( \Delta_i \). Now consider the following calculations:

\[
U(\lambda \otimes \pi(x)) U^*(H) = U \lambda \otimes \pi(x) \left( \sum_{i \in I} \langle H(\cdot), e_i \rangle \otimes e_i \right)
\]

\[
= U \left( \sum_{i \in I} \langle H(x^{-1} \cdot), e_i \rangle \otimes \pi(x) e_i \right)
\]

\[
= \pi(x) H(x^{-1} y)
\]

\[
= \Delta_1(x) H(y)
\]

\[
W^* \Delta_1(x) WH(y) = W^* \Delta_1(x) \pi(x) H(y)
\]
\[ \Delta_2(x)H(y) = H(x^{-1}y) \]

\[ U^* \Delta_2(x)U(\sum_{j \in I} f_j \otimes e_j)(y) = U^* \Delta_2(x) \sum_{j \in I} f_j(y) e_j \]

\[ = U^* \sum_{j \in I} \langle f_j(x^{-1}y)e_j|e_i \rangle \otimes e_i \]

\[ = \sum_{i \in I} f_i(x^{-1}y) \otimes e_i \]

\[ = \lambda(x) \otimes I(\sum_{j \in I} f_j \otimes e_j)(y) \]

\[ = \Delta_3(x)(\sum_{j \in I} f_j \otimes e_j)(y) \]

\[ V^* \Delta_3(x)V(f_j)_{j \in I} = V^* \Delta_3(x) \sum_{j \in I} f_j \otimes e_j \]

\[ = V^* \sum_{j \in I} \lambda(x) f_j \otimes e_j \]

\[ = (\lambda(x)f_j)_{j \in I} \]

\[ = I \cdot \lambda(x)(f_j)_{j \in I} \]

**Corollary 3.4.** \( A(G) \) is an ideal of \( B(G) \)

**Proof.** Let \( \pi_{\zeta,\eta} \in B(G) \) and \( u \in A(G) \). By Theorem 3.2, we know that \( u \) is of the form \( u = \sum_{j=1}^{\infty} \lambda f_j g_j \in A(G) \). An application of Fell’s Absorption
Principle yields that:

\[
\pi_{\zeta,\eta} u(x) = \sum_{j=1}^{\infty} \langle \lambda \otimes \pi(x) f_j \otimes \zeta, j_j \otimes \eta \rangle \\
= \sum_{j=1}^{\infty} \langle L^* I \cdot \lambda(x) L f_j \otimes \zeta, g_j \otimes \eta \rangle \\
= \sum_{j=1}^{\infty} \langle (\lambda(x) f_{j,i}), (g_{i,j}) \rangle_{i \in I} \\
= \sum_{j=1}^{\infty} \sum_{i \in I} \langle \lambda(x) f_{j,i}, g_{j,i} \rangle \in A(G)
\]

It turns out that \( A(G) \) possesses a regularity property that will prove to be of fundamental importance.

**Lemma 3.5 (Eymard’s Trick).** Let \( U \subseteq G \) be an open set, and let \( K \subseteq U \) be a compact set. Then there exists \( u \in A(G) \) such that \( u|_K = 1, u_{U^c} = 0 \).

**Proof.** Let \( V \) be a relatively compact neighborhood of \( e \) such that \( KV^2 \subseteq U \), the existence of which is guaranteed by Proposition 2.2. Let \( 1_V \) and \( 1_{KV} \) denote the characteristic functions for \( V \) and \( KV \) respectively. Then consider the function

\[
u(x) = \frac{1}{m(V)} \lambda_{1_V,1_{KV}}
\]

It is immediate that \( u \in A(G) \), and a simple calculation yields that

\[
u(x) = \frac{1}{m(V)} \int_G 1_V(x^{-1}y)1_{KV}(y)dy = \frac{1}{m(V)} \int_G 1_{xV \cap KV}(y)dy = \frac{m(xV \cap KV)}{m(V)}
\]
If $x \in K$ by the left-invariance of Haar measure we have that $u(x) = \frac{m(xV)}{m(V)} = 1$. Conversely, if $x \notin U$ then $xV \cap KV = \emptyset$ so $u(x) = \frac{m(\emptyset)}{m(V)} = 0$.

**Remark 2.** Note that if we let $U = G$, an important consequence of this lemma is that for any compact set we can always find a function in $A(G)$ that is constantly one on that set. This is actually quite useful, because in general $A(G)$ will not be a unital algebra. After all, $A(G)$ is an ideal in $B(G)$ so if $1 \in A(G)$ then $A(G) = B(G)$. As will be addressed later in this section, it turns out that this is only possible if $G$ is a compact group.

**Lemma 3.6.** Let $u = \lambda_{f,g} \in A(G)$. Then $\|u\|_{A(G)} \leq \|f\|_2\|g\|_2$

*Proof.* Observe that if $u = \lambda_{f,g}$ then

$$\|u\|_{A(G)} = \sup \{| \int_G h(x)u(x)dx | : h \in L^1(G), \|h\|_* \leq 1\}$$

$$= \sup \{\langle \lambda(h)f, g \rangle : h \in L^1(G), \|h\|_* \leq 1\}$$

$$\leq \|f\|_2\|g\|_2$$

**Theorem 3.7.** $\overline{\text{span}\{\lambda_{f,g} : f, g \in C_C(G)\}}^{\|\cdot\|_{B(G)}} = B(G) \cap C_C(G) = A(G)$

*Proof.* Take $v \in B(G) \cap C_C(G)$. Then $\text{supp}(v)$ is compact, so by Eymard’s Trick we can find $u \in A(G)$ such that $u|_{\text{supp}(v)} = 1$. Then because $A(G)$ is an ideal of $B(G)$ it follows that $uv = v \in A(G)$, which means that $B(G) \cap C_C(G) \subseteq A(G)$.

Because clearly $\text{span}\{\lambda_{f,g} : f, g \in C_C(G)\} \subseteq B(G) \cap C_C(G)$, then it suffices to show that $\overline{\text{span}\{\lambda_{f,g} : f, g \in C_C(G)\}}^{\|\cdot\|_{B(G)}} = A(G)$. Let $\lambda_{f,g} \in A(G)$ with $f, g \in L^2(G)$, and $\epsilon > 0$ be given. Then by the density of $C_C(G)$ in $L^2(G)$, we
can find $h, k \in C_C(G)$ such that $\|f - h\|_2, \|g - k\|_2 < \epsilon$. Then by Lemma 3.6 we have that

$$
\|\lambda_{f,g} - \lambda_{h,k}\|_{B(G)} = \|\lambda_{f-h,g} - \lambda_{h,g-k}\|_{B(G)} \\
\leq \|f - h\|_2 \cdot \|g\|_2 + \|h\|_2 \cdot \|g - k\|_2 \\
\leq \epsilon \cdot \|g\|_2 + \epsilon (\epsilon + \|f\|_2)
$$

Thus by letting $\epsilon$ get small our result follows.

\[\square\]

**Remark 3.** Because $A(G) \subseteq B(G)$ and $B(G) \cap C_C(G) \subseteq A(G)$, then it follows that $B(G) \cap C_C(G) = A(G) \cap C_C(G)$ so our above result is really telling us that $\overline{A(G) \cap C_C(G)}_{\|\cdot\|_{A(G)}} = A(G)$

**Corollary 3.8.** $A(G)$ is uniformly dense in $C_0(G)$

**Proof.** First note that if $u \in A(G)$ then

$$
\|u\|_{A(G)} = \sup \left\{ \int_G u(s)f(s)ds | f \in L^1(G), \|f\|_* \leq 1 \right\} \\
\geq \sup \left\{ \int_G u(s)f(s)ds | f \in L^1(G), \|f\|_1 \leq 1 \right\} \\
= \|u\|_{\infty}
$$

Thus because $A(G) \cap C_C(G)$ is dense in $A(G)$ it follows that $A(G) \subseteq C_0(G)$. Now it is clear that $A(G)$ is self-adjoint and by Eymard’s Trick we have that $A(G)$ is point separating, hence by the Stone-Weierstrass Theorem we have that $A(G)$ is dense in $C_0(G)$ under the uniform norm.

\[\square\]

We will end this section by seeing what we can say about $A(G)$ if we restrict our focus to specific classes of groups.

**Proposition 3.9.** $G$ is compact if and only if $A(G) = B(G)$
3.2 The Fourier Algebra

**Proof.** Suppose that $G$ is compact. We have that by Theorem 3.7 that $B(G) \cap C_c(G) \subseteq A(G)$, but because $C_c(G) = C(G)$ then we in fact have that $B(G) \subseteq A(G)$, which can only be possible if $A(G) = B(G)$.

Conversely, suppose that $A(G) = B(G)$. Now Corollary 3.8 tells us that $A(G) \subseteq C_0(G)$, so because $A(G) = B(G)$ then it follows that $1 \in C_0(G)$. This of course is only possible if $G$ is compact. 

**Proposition 3.10.** If $G$ is abelian, then $A(G) \cong L^1(\hat{G})$

**Proof.** By Theorem 3.2 we know that $A(G)^* \cong VN(G)$, and of course it is well-known that $L^1(\hat{G})^* \cong L^\infty(\hat{G})$. Because the predual of a von Neumann algebra is unique, it suffices to show that $VN(G) \cong L^\infty(\hat{G})$.

Consider the representation $\hat{\lambda} : G \to U(L^2(\hat{G}))$ defined by $\hat{\lambda}(x)f(\sigma) = \sigma(x)f(\sigma)$. Let $U$ be the unitary given from Plancherel’s Theorem and choose any $x \in G, f \in L^1 \cap L^2(\hat{G})$. Then

$$U\lambda(x)U^*f(\sigma) = \int_G U^*f(x^{-1}y)\overline{\sigma(y)}dm(y)$$
$$= \int_G U^*f(y)\overline{\sigma(xy)}dm(y)$$
$$= \sigma(x)\int_G U^*f(y)\overline{\sigma(y)}dm(y)$$
$$= \overline{\sigma(x)U(U^*f)\sigma}$$
$$= \sigma(x)U(U^*f)\sigma $$

Because $L^1 \cap L^2(\hat{G})$ is dense in $L^1(\hat{G})$, then it follows that $\lambda$ and $\hat{\lambda}$ are unitarily equivalent. An application of the Hahn-Banach theorem yields that $\overline{\text{span} \lambda(G)}^{w*} \cong L^\infty(\hat{G})$. Because $\hat{\lambda}$ and $\lambda$ are unitarily equivalent, then we have
that $VN(G) \cong L^\infty(\hat{G})$, as desired.

\[\square\]

4 The Spectrum of $A(G)$

As mentioned in the introduction, we want to think of $A(G)$ as representing some sort of generalization of Pontrayagin duality. One way of testing if $A(G)$ performs an adequate job at this task is if we can recover any sort of group structure from $A(G)$, which it turns out that we can. But first, we need a technical lemma which tells us that singletons are sets of spectral synthesis for $A(G)$.

**Lemma 4.1.** Let $a \in G, f \in A(G)$ such that $f(a) = 0$. Then given $\epsilon > 0$, there exists $h \in A(G) \cap C_c(G)$ vanishing on a neighborhood of $a$ such that $\|h - f\|_{A(G)} < \epsilon$.

**Proof.** Without loss let $\epsilon < 1$. Because $A(G) \cap C_c(G)$ is dense in $A(G)$ by Theorem 3.7, we can find $f' \in A(G) \cap C_c(G)$ such that $\|f - f'\|_{A(G)} < \epsilon$. As noted in Corollary 3.8, we have that $\|\cdot\|_{A(G)} \geq \|\cdot\|_{\infty}$, so $\|f - f'\|_{A(G)} < \epsilon$. In particular, this means we can find a relatively compact open neighborhood $V'$ of $e$ such that $\sup\{|f'(ay)| : y \in V'\} < \epsilon$. Define the set

$$W = \{y \in G : \|f' - R_y f'\|_{A(G)} < \epsilon\}.$$

We have that $W$ is a neighborhood of $e$ by the continuity of translation, so if we set $V = V' \cap W$ then $\sup\{|f'(ay)| : y \in V\} < \epsilon$. Then by the regularity of Haar measure, there exists a compact neighborhood $K$ of $e$ with $K \subseteq V$ and
Now define the following functions:

\[ u = \frac{1}{m(K)}1_K \]

\[ g = 1_{aV}f' \]

\[ h(x) = (f' - g) \ast \tilde{u} = \frac{1}{m(K)}\int_K f'(xy)[1 - 1_{aV(xy)}]dy \in A(G) \]

All of the above functions have compact support because \( f' \) has compact support and \( V \) has compact closure. Now pick an open neighborhood \( O \) of \( a \) such that \( OK \subseteq aV \). It follows that \( h = 0 \) on a neighborhood of \( a \). Now observe the following calculations:

\[ \|u\|_2 = \left( \int_K \left( \frac{1}{m(K)} \right)^2 \right)^{\frac{1}{2}} \leq \frac{1}{m(K)^{\frac{1}{2}}} \left( \frac{1}{1 - \epsilon} \right)^{\frac{1}{2}} \]

\[ \|g\|_2 = \left( \int_{aV} |f'(y)|^2 dy \right)^{\frac{1}{2}} \leq \epsilon m(V)^{\frac{1}{2}} \]

\[ \|f' - f' \ast \tilde{u}\|_{A(G)} = \|f' - \frac{1}{m(K)}\int_K R_yf' dy\|_{A(G)} \]

\[ = \left\| \frac{1}{m(K)}\int_K f'(y) - R_y f' dy \right\|_{A(G)} \]

\[ \leq \sup_{y \in K} \|f' - R_yf'\|_{A(G)} \]

\[ < \epsilon \]

Then with all of these calculations in hand, an application of Lemma 3.6 yields that

\[ \|f - h\|_{A(G)} \leq \|f - f'\|_{A(G)} + \|f' - h\|_{A(G)} \]
\[ \leq \epsilon + \| f' - f' \ast \tilde{u} \|_{A(G)} + \| g \ast \tilde{u} \|_{A(G)} \]
\[ < 2\epsilon + \| g \|_2 \| u \|_2 \]
\[ < 2\epsilon + \epsilon \left( \frac{1}{1 - \epsilon} \right)^{\frac{1}{2}} \]

\[ \square \]

**Definition 4.2.** If \( \mathcal{A} \) is a commutative algebra, let \( \sigma(\mathcal{A}) = \{ \phi : \mathcal{A} \to \mathbb{C} : \phi \) a nonzero algebra homomorphism\} denote the spectrum of \( \mathcal{A} \).

**Theorem 4.3.** \( G \) can be identified with \( \sigma(A(G)) \) via the mapping \( x \mapsto \phi_x \), where \( \phi_x \) is point evaluation at \( x \in G \).

**Proof.** First, we note that the above map is clearly injective because \( A(G) \) separates points by Eymard’s Trick. Now let us suppose that the map is not surjective, that is, there exists \( \phi \in \sigma(A(G)) \) such that \( \phi \) is not point evaluation. Then for any \( x \in G, \phi \neq \phi_x \). We can find \( u_x \in A(G) \) such that \( \phi(u_x) \neq 0 \) and \( \phi_x(u_x) = 0 \). By scaling we can assume that in fact \( \phi(u_x) = 1 \). From the above lemma we can approximate \( u_x \) with a sequence from \( A(G) \cap C_{C^*}(G) \) each vanishing on a neighborhood of \( x \), so again without loss assume that \( u_x \) vanishes on an open neighborhood \( V_x \) of \( x \). Then by Theorem 3.7, we can find \( u_0 \in A(G) \cap C_{C^*}(G) \) such that \( \phi(u_0) = 1 \) (once again scaling if necessary). Now \( \{ V_x \}_{x \in G} \) is an open cover of \( \text{supp}(u_0) \), so by compactness we can find \( x_1, \ldots, x_n \in \text{supp}(u_0) \) such that \( \text{supp}(u_0) \subseteq \bigcup_{j=1}^n V_{x_j} \). Set the function

\[ u = u_0 \prod_{j=1}^n u_{x_j} \in A(G) \]

For \( x \in G \) if \( x \in \text{supp}(u_0) \) then we can find \( x_i \) such that \( x \in V_{x_i} \), so \( u_{x_i}(x) = 0 \). Meanwhile, by definition we have that \( u_0|_{\text{supp}(u)} = 0 \), so it follows that \( u = 0 \).
However, note that
\[ \phi(u) = \phi(u_0) \prod_{j=1}^{n} \phi(u_{x_j}) = 1 \]
which creates a contradiction because \( u = 0 \). Thus \( \phi = \phi_x \) for some \( x \), as desired. \( \square \)

5 Leptin’s Theorem

Recall from our discussion in Section 3.2 that knowing certain facts about the structure of \( G \), such as whether it is abelian or compact, allows us to know more about the structure of \( A(G) \). In this section we explore the more general case where \( G \) is amenable.

**Definition 5.1.** Let \( A \) be a commutative normed algebra. Then an approximate identity for \( A \) is a net \( (e_\alpha)_{\alpha \in I} \subseteq A \) such that \( \lim_{\alpha} \|e_\alpha a - a\| = 0 \) for all \( a \in A \). We call \( (e_\alpha)_{\alpha \in I} \) a bounded approximate identity if in addition it is bounded as a net, that is, there exists \( c \in \mathbb{R} \) such that \( \|e_\alpha\| \leq c \) for all \( \alpha \in I \).

**Remark 4.** Let \( \{U_\alpha\}_\alpha \) be the neighborhood basis of \( e \) consisting of relatively compact sets. Then it turns out that the net \( \left\{ \frac{1}{m(U_\alpha)}1_{U_\alpha} \right\}_\alpha \) is a bounded approximate identity for \( \ell^1(G) \).

**Remark 5.** It is not the case that all Banach algebras possess approximate identities that are bounded. For example, consider \( \ell^1(\mathbb{N}) \). Let \( a = \sum_{i=1}^{\infty} a(i)\delta_i, b = \sum_{i=1}^{\infty} b(i)\delta_i \in \ell^1(\mathbb{N}) \) and define multiplication by
\[ a \cdot b = \left( \sum_{n=1}^{\infty} a(n)\delta_n \right) \left( \sum_{n=1}^{\infty} b(n)\delta_n \right) = \sum_{n=1}^{\infty} a(n)b(n)\delta_n \]
Then it is easy to see that if
\[ e_n = \sum_{i=1}^{n} \delta_n \]
then \((e_n)_{n \in \mathbb{N}}\) is an approximate identity for \(\ell^1(\mathbb{N})\) such that \(\|e_n\|_1 = n\), so \((e_n)_{n \in \mathbb{N}}\) is an unbounded approximate identity. Observe that if \((e_\alpha)_{\alpha \in I}\) is any approximate identity for \(\ell^1(\mathbb{N})\) then if \(n \in \mathbb{N}\) it follows that
\[
\limsup_{\alpha \in I} \|e_\alpha\|_1 \geq \lim_{\alpha \in I} \sum_{i=1}^{n} |e_\alpha(i)| = n.
\]
Thus we can see that \(\sup_{\alpha \in I} \|e_\alpha\|_1 \geq \infty\), so \((e_\alpha)_{\alpha \in I}\) also must be unbounded. Hence \(\ell^1(\mathbb{N})\) is a Banach algebra with only unbounded approximate identities.

**Lemma 5.2.** If \(f \in L^1(G)\) only attains nonnegative real values, then \(\|f^{*n}\|_1 = \|f\|_1^n\), where \(f^{*n}\) denotes the convolution of \(f\) with itself \(n\) times.

**Proof.** It suffices to prove the result for \(n = 2\), and let the rest follow by induction. In order to achieve this, we will confirm that the functional \(f \mapsto \int_G f\) is multiplicative with respect to convolution. By Fubini’s theorem and the fact that the Haar integral is left-invariant, we have the following:

\[
\int_G f * g = \int_G \int_G f(x)g(x^{-1}y) dxdy
= \int_G f(x) \int_G g(x^{-1}y) dydx
= \int_G f(x) \int_G g(y) dydx
= \int_G f \int_G g
\]

Because \(f, g \geq 0\) a.e., it follows that
\[
\|f * g\|_1 = \int_G f * g = \int_G f \int_G g = \|f\|_1 \|g\|_1.
\]
Remark 6. Observe that if we allow ourselves to consider a general \( f \in L^1(G) \) then above proof easily extends to show that \( \|f^*\|_1 \leq \|f\|_1 \), which is why \( L^1(G) \) is a Banach algebra with convolution as multiplication.

Theorem 5.3. (Leptin's Theorem) A locally compact group \( G \) is amenable if and only if \( A(G) \) has a bounded approximate identity.

Proof. Suppose that \( G \) is amenable. Let \( K \subseteq G \) be compact and \( \epsilon > 0 \). Then by Proposition 2.5 we can find a compact set \( C \) such that \( m(KC) < (1 + \epsilon)m(C) \). Now let us consider the following collection of functions:

\[
u_{K,\epsilon}(x) = \frac{1}{(1 + \epsilon)m(C)} \lambda_{1_{KC},1_C}(x) = \frac{1}{(1 + \epsilon)m(C)} \int_C 1_{KC}(xy)dy
\]

Clearly by construction each \( u_{K,\epsilon} \in A(G) \cap C_C(G) \). If we let \( \mathcal{K}(G) \) denote the set of all compact subsets of \( G \), then we can define a preorder on the set \( \mathcal{K}(G) \times \mathbb{R}^+ \) by letting \( (K, \epsilon) < (K', \epsilon') \) if \( K \subseteq K' \) and \( \epsilon < \epsilon' \). This allows us to consider \( (u_{K,\epsilon})_{K,\epsilon} \) as a net. By the following calculation, we have that

\[
\|u_{K,\epsilon}\|_{A(G)} \leq \frac{\|1_C\|_2 \|1_{KC}\|_2}{(1 + \epsilon)m(C)}m(C) = \frac{m(KC)^{1/2}}{(1 + \epsilon)m(C)^{1/2}} \leq 1
\]

Thus we have that \( (u_{K,\epsilon})_{K,\epsilon} \) is in fact a bounded net. Now let \( v \in A(G) \cap C_C(G) \) and \( \epsilon > 0 \). Set \( K = \text{supp}(v) \). Observe that because \( u_{K,\epsilon}|_K = \frac{1}{1 + \epsilon} \) then it follows that \( u_{K,\epsilon}v(x) = \frac{v(x)}{1 + \epsilon} \). We then have that

\[
\|u_{K,\epsilon}v - v\|_{A(G)} = \frac{\epsilon}{1 + \epsilon}\|v\|_{A(G)}
\]

Thus we have that \( \lim_{(K,\epsilon)}\|u_{K,\epsilon}v - v\|_{A(G)} = 0 \) for all \( v \in A(G) \cap C_C(G) \). This is not quite yet enough to satisfy the condition of being an approximate
identity because this condition needs to hold for all elements of $A(G)$, not just a subset. Fortunately, it turns out the density of $A(G) \cap C_C(G)$ in $A(G)$ carries us the rest of the way. To see that this is true, let $w \in A(G)$ and $\epsilon > 0$. Then by density there exists $v \in A(G) \cap C_C(G)$ such that $\|w - v\|_{A(G)} < \epsilon$. Set $K = \text{supp}(v)$. Then we have that

$$\|u_{K,\epsilon}w - w\|_{A(G)} = \|u_{K,\epsilon}w - u_{K,\epsilon}v + u_{K,\epsilon}v - v + v - w\|_{A(G)} \leq \|u_{K,\epsilon}w - u_{K,\epsilon}v\|_{A(G)} + \|u_{K,\epsilon}v - v\|_{A(G)} + \|v - w\|_{A(G)} \leq \|u_{K,\epsilon}v - v\|_{A(G)} + 2\epsilon$$

From our work above we know that we can allow the above quantity to get arbitrarily small, hence we have our desired result that $(u_{K,\epsilon})_{K,\epsilon}$ is a bounded approximate identity for $A(G)$.

Conversely, suppose that $(u_\alpha)_\alpha$ is a bounded approximate identity for $A(G)$ with bound $c$. We wish to show that Proposition 2.5(ii) is satisfied. Let $f \in C_C^+(G)$ and set $K = \text{supp}(f)$. By Eymard’s Trick we can find $u \in A(G)$ such that $u|_K = 1$. We have that $\lim_\alpha \|u_\alpha u - u\|_{A(G)} = 0$, so because $\|\cdot\|_{\infty} \leq \|\cdot\|_{A(G)}$ then it follows that $\lim_\alpha \|u_\alpha u - u\|_{\infty} = 0$. Because $u|_K = 1$, then we have that $u_\alpha$ converges uniformly to $1$ on $K$. Thus if $\epsilon > 0$ we can find $\alpha_\epsilon$ such that $\text{Re}(u_{\alpha_\epsilon}) \geq 1 - \epsilon$ for all $x \in K$. It then follows that

$$\text{Re}\langle u_{\alpha_\epsilon}, f \rangle = \int_G \text{Re}(u_{\alpha_\epsilon}(x)f(x))dx \geq (1 - \epsilon)\|f\|_1$$
5 LEPTIN’S THEOREM

However, also observe that

$$|\langle u_\alpha, f \rangle| \leq \|\lambda(f)\| \cdot \|u_\alpha\|_{A(G)} \leq c\|\lambda(f)\|$$

Because $\epsilon$ is allowed to vary, then it must hold that $\|f\|_1 \leq c\|\lambda(f)\|$. Then by Lemma 5.2, we have that

$$\|f\|_1^n = \|f^{*n}\|_1 \leq c\|\lambda(f^{*n})\| \leq c\|\lambda(f)\|^n$$

By taking limits on $n$ we can see that

$$\|f\|_1 \leq \|\lambda(f)\| \lim_{n \to \infty} c^\frac{1}{n} = \|\lambda(f)\| \leq \|f\|_1$$

Thus by Proposition 2.5 we have that $G$ is amenable. \qed

**Remark 7.** It is of interest to note that the above proof actually implies that if $A(G)$ has a bounded approximate identity, then it has an approximate identity that is bounded by 1 (we call such nets *contractive*). This is most certainly not true for all Banach algebras. Let $S$ be a finite semigroup, and consider $\ell^1(S)$. If $a = \sum_{s \in S} a(s)\delta_s, b = \sum_{s \in S} b(s)\delta_s \in \ell^1(S)$ then we define multiplication between $a$ and $b$ by

$$a \ast b = \sum_{s \in S} \sum_{t \in S} a(s)b(t)\delta_{st} = \sum_{u \in S} (\sum_{st = u} a(s)b(s))\delta_u.$$ 

Notice that in particular that $\delta_s \ast \delta_t = \delta_{st}$. Furthermore, we can see that

$$\|a \ast b\|_1 = \sum_{u \in S} |\sum_{st = u} a(s)b(s)| \leq \sum_{u \in S} (\sum_{st = u} |a(s)||b(s)|) = \sum_{s \in S} |a(s)| \sum_{t \in S} |b(t)| = \|a\| \|b\|_1$$

Hence it follows that $\ell^1(S)$ is a Banach algebra. Now let us consider the abelian
semigroup $S_n = \{s_1 \ldots, s_n, o\}$ where semigroup multiplication is defined as

$$s_j s_k = \begin{cases} s_j & \text{if } j = k \\ o & \text{otherwise,} \end{cases} \quad os_j = o = s_j o = oo$$

Consider the element

$$e = \sum_{j=1}^{n} \delta_{s_j} - (n-1)\delta_o$$

We can see that if $1 \leq j \leq n$ then

$$\delta_{s_j} e = (e(s_j))\delta_{s_j} = \delta_{s_j}$$

and

$$\delta_o e = (\sum_{i=1}^{n} e(s_j) + e)s_o))\delta_o = (n - (n - 1))\delta_o = \delta_o.$$

Hence it follows that $e$ is an identity for $\ell^1(S_n)$. An identity is trivially an approximate identity, however note that $\|e\|_1 = 2n - 1$ so $e$ is a bounded approximate identity that is not contractive.

Now let $(e_\alpha)_{\alpha \in I}$ be a bounded approximate identity for $\ell^1(S_n)$. Because $B_r(\ell^1(S_n))$ is compact, where $r = \sup_\alpha \|e_\alpha\|$, then we can find a cluster point $e'$ for $(e_\alpha)_{\alpha \in I}$. By passing to an appropriate subnet we can see that

$$\|e'\|_1 \leq r \text{ and } \|e' - e\|_1 = \|e e' - e\|_1 = \lim_{\beta} \|e e_{\alpha(\beta)} - e\|_1 = 0$$

It follows that $e' = e$, so $e'$ is unique. Thus we have that $r \geq \|e\|_1 = 2n - 1$. Therefore we have that $(e_\alpha)_{\alpha \in I}$ is not contractive, so we have shown that $\ell^1(S_n)$ is a Banach algebra with bounded approximate identities but no contractive
approximate identities.

For another example that is more directly related to the material in this project, if $H$ is a closed subgroup of $G$ that is not open, then according to [6] the ideal $\{u \in A(G) : u(h) = 0 \text{ for all } h \in H\}$ contains an approximate identity bounded by 2, and in fact it turns out that no number strictly less than 2 can act as a suitable bound for any approximate identities on this algebra.

6 Restriction Map to Subgroups

Recall that $A(G)$ is the predual of $VN(G)$ through the dual pairing $\langle \lambda_{f,g}, T \rangle = \langle Tf, g \rangle$. If $u \in A(G)$ and $T \in VN(G)$, we can define an action $T \cdot u$ by

$$\langle T \cdot u, S \rangle = \langle u, ST \rangle \text{ where } S \in VN(G)$$

Let $u = \sum_{i=1}^{\infty} \lambda_{f_i,g_i} \in A(G)$. Then it follows that

$$u(t) = \sum_{i=1}^{\infty} \langle \lambda(t)f_i, g_i \rangle$$

$$= \sum_{i=1}^{\infty} \langle \lambda_{f_i,g_i}, \lambda(t) \rangle$$

$$= \langle u, \lambda(t) \rangle$$

We require the following useful lemma

**Lemma 6.1.** Let $u = \sum_{i=1}^{\infty} \lambda_{f_i,g_i} \in A(G)$.

(i) If $T \in VN(G)$, then $T \cdot u = \sum_{i=1}^{\infty} \lambda_{Tf_i,g_i}$

(ii) If $f \in L^1(G)$, then $\lambda(f) \cdot u = (f * \check{u})$
Proof. (i) Let $S \in VN(G)$. Then we calculate

$$
\langle T \cdot u, S \rangle = \langle u, ST \rangle
$$

$$
= \langle \sum_{i=1}^{\infty} \lambda_{f_i, g_i}, ST \rangle
$$

$$
= \sum_{i=1}^{\infty} \langle \lambda_{f_i, g_i}, ST \rangle
$$

$$
= \sum_{i=1}^{\infty} \langle ST f_i, g_i \rangle
$$

$$
= \sum_{i=1}^{\infty} \langle \lambda_{T f_i, g_i}, S \rangle
$$

$$
= \langle \sum_{i=1}^{\infty} \lambda_{T f_i, g_i}, S \rangle
$$

Hence it follows that $T \cdot u = \sum_{i=1}^{\infty} \lambda_{T f_i, g_i}$.

(ii) Let $s \in G$. Then by the above remark, we have that

$$
\lambda(f) \cdot u(s) = \langle \lambda(f) \cdot u, \lambda(s) \rangle
$$

$$
= \langle u, \lambda(f) \lambda(s) \rangle
$$

$$
= \langle u, \lambda(\lambda(s)f) \rangle
$$

$$
= \int_G f(s^{-1}t)u(t)dt
$$

$$
= \int_G f(t)u(st)dt
$$

$$
= \int_G f(t)\hat{u}(t^{-1}s^{-1})dt
$$

$$
= (f \ast \hat{u})(s^{-1})
$$
Thus we have that \( \lambda(f) \cdot u = (f \ast \tilde{u}) \)

\[ \]  

**Lemma 6.2.** Let \( h \in C_C(G) \). Then the operator \( C_h : L^2(G) \to L^2(G) \) defined by \( f \mapsto f \ast h \) for \( f \in L^2(G) \) commutes with every other operator in \( VN(G) \).

**Proof.** First we will show that \( C_h \) is bounded. Define the representation \( \sigma : G \to U(L^2(G)) \) by \( \sigma(s)f(t) = \frac{1}{\Delta(s)^{\frac{1}{2}}} f(ts) \), which we will call the right regular representation.

Observe that for \( f, g, k \in L^2(G) \) then by Fubini’s Theorem and properties of Haar measure we have that

\[
\langle \sigma(f)g, k \rangle = \int_G f(s) \int_G \frac{1}{\delta(s)^{\frac{1}{2}}} g(ts) \overline{k(t)} dtds
= \int_G \int_G f(s) \frac{1}{\Delta(s)^{\frac{1}{2}}} g(ts) \overline{k(t)} dsdt
= \int_G \int_G f(t^{-1}s) \frac{1}{\Delta(t^{-1}s)^{\frac{1}{2}}} g(s) \overline{k(t)} dsdt
= \int_G \int_G g \ast (\Delta^{\frac{1}{2}} \tilde{f})(t) \overline{k(t)} dt
= \langle g \ast (\Delta^{\frac{1}{2}} \tilde{f}), k \rangle
\]

Because this holds for all \( k \in L^2(G) \), it follows that \( \sigma(f)g = g \ast (\Delta^{\frac{1}{2}} \tilde{f}) \). Now clearly \( \Delta^{\frac{1}{2}} \tilde{f} \in C_C(G) \subseteq L^1(G) \), hence we have that \( \|C_h\| = \|\sigma(\Delta^{\frac{1}{2}} \tilde{f})\| \leq \|\Delta^{\frac{1}{2}} \tilde{f}\|_1 \), so it follows that \( C \) is bounded operator.

Because \( VN(G) = \text{span} \lambda(L^1(G))^{\ast \ast} \), by continuity it suffices to show that \( C_h \) commutes with \( \lambda(f) \) for all \( f \in L^1(G) \). Then for \( g \in L^2(G) \), we observe that

\[
C_h \lambda(f)g = C_h(f \ast g) = f \ast g \ast h = \lambda(f)(g \ast h) = \lambda(f) C_h g
\]

\[ \]
Theorem 6.3. Let $H$ be a closed subgroup of $G$. Then $A(G)|_H = A(H)$

Proof. First we must confirm that $A(G)|_H \subseteq A(H)$. Let $\sigma : H \rightarrow G$ denote the inclusion map. Then $\lambda_G \circ \sigma$ is a continuous unitary representation of $H$, and using the characterization of $A_{\lambda_G \circ \sigma}$ from Theorem 3.2, then we can easily see that $A(G)|_H = A_{\lambda_G \circ \sigma}$, hence $A(G)|_H$ is a closed subspace in $B(H)$. Furthermore, if $u \in A(G) \cap C_C(G)$ then certainly $u|_H \in B(H) \cap C_C(H) \subseteq A(H)$ by Theorem 3.7, and furthermore the density which is guaranteed by that theorem implies that $A(G)|_H \subseteq A(H)$.

We claim that $A(G)|_H$ is invariant under the action of $VN(H)$ on $A(H)$. Let $s, t \in H$ and $u \in A(G)$. Then

$$
\lambda(s) \cdot u(t) = \langle \lambda(s) \cdot s, \lambda(t) \rangle = \langle u, \lambda(t) \lambda(s) \rangle = \langle u, \lambda(ts) \rangle = u(ts)
$$

Thus we have that $\lambda(s) \cdot u \in A(G)|_H$, so it follows that $T \cdot u \in A(G)|_H$ for all $T \in \text{span}\lambda(H)$. Because $VN(H) = \text{span}(VN(H))_{sa}$ by von Neumann’s Double Commutant Theorem, we only need to show that the result holds for $T \in B_1(VN(H))_{sa}$. For such a $T$, by Kaplansky’s Density Theorem we have that $B_1(VN(H))_{sa} = \overline{\text{span}\lambda(H)}_{sa}^{\text{SOT}}$ because $VN(H) = \overline{\text{span}\lambda(H)}^{\text{SOT}}$, hence we can find a net $\{T_\alpha\}_\alpha \subseteq B_1(\text{span}\lambda(H))$ converging to $T$ in the strong operator topology. Let $u = \sum_{i=1}^{\infty} \lambda_{f_i, g_i} \in A(H)$ and $\epsilon > 0$. Because $\sum_{i=1}^{\infty} \|f_i\|_2 \|g_i\|_2 < \infty$ we can find an $n_\epsilon \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} \|f_i\|_2 \|g_i\|_2 < \frac{\epsilon}{4}$. Also, by strong operator convergence we know that for $1 \leq i \leq n$ we can find $\alpha_i$ such that for all $\alpha \geq \alpha_i$, then $\|T f_i - T_{\alpha} f_i\| < \frac{\epsilon}{2 \sup\{\|g_j\|_2 : 1 \leq j \leq n\}}$ (note that we can always pick a large enough $n$ such that this supremum is nonzero; otherwise the result is trivial). Set $\alpha_\epsilon$ such that $\alpha_\epsilon \geq \alpha_i$ for all $1 \leq i \leq n$. Let $\alpha \geq \alpha_\epsilon$
and observe that
\[ \|T \cdot u - T_\alpha \cdot u\|_{A(H)} = \| \sum_{i=1}^{\infty} \lambda_T f_i, g_i - \sum_{i=1}^{\infty} \lambda_{T_\alpha f_i, g_i} \|_{A(H)} \]
\[ = \| \sum_{i=1}^{\infty} \lambda_{(T-T_\alpha)f_i, g_i} \|_{A(H)} \]
\[ \leq \sum_{i=1}^{n} \|T f_i - T_\alpha f_i\|_2 \|g_i\|_2 + 2 \sum_{i=n+1}^{\infty} \|f_i\|_2 \|g_i\|_2 \]
\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} \]
\[ = \epsilon \]

Thus we have that \( T \cdot u = \lim_\alpha T_\alpha \cdot u \), but because \( A(G)|_H \) is closed in \( A(H) \) it follows that \( T \cdot u \in A(G)|_H \), as desired.

Now suppose that \( A(G)|_H \neq A(H) \). Then the Hahn-Banach Theorem yields that there must exist some \( T \in VN(H) - \{0\} \) such that \( \langle u, T \rangle = 0 \) for all \( u \in A(G)|_H \).

By the density guaranteed from Theorem 3.7, we can find some \( h \in \text{span} \{ \lambda_{f,g} : f, g \in C_C(G) \} \) such that \( Th \neq 0 \) because \( T \neq 0 \). By Eymard’s Trick we find a function \( u \in A(G) \cap C_C(G) \) such that \( u|_H = 1 \), thus it follows that the product \( uTh \neq 0 \). Then if \( v \in A(G)|_H \) we have by Lemma 6.1 and Lemma 6.2 that
\[ \langle \lambda(h) \cdot uv, T \rangle = \langle (h \ast \check{uv}), T \rangle \]
\[ = T(h \ast \check{uv})(e) \]
\[ = (Th) \ast \check{uv}(e) \]
\[ = \int_G u(s)Th(s)v(s)ds \]
In particular, because $A(G)|_H$ is invariant under the action of $T$ then it follows that $\int_G u(s)Th(s)v(s)ds = 0$. The Stone-Weierstrass Theorem yields that $\overline{A(G)|_H}_{\|\cdot\|_{\infty}} = \mathcal{C}_0(H)$ hence it follows that $\int_G u(s)Th(s)w(s)ds = 0$ for all $w \in \mathcal{C}_0(H)$. Thus $uTh = 0$, which is a contradiction of our choice of $T$. Therefore it must be true that $A(G)|_H = A(H)$, as desired. \qed
References


