On $S$-unit equations in two unknowns

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\S 0. Introduction

Let $K$ be an algebraic number field of degree $d$, with discriminant $D_K$ and ring of integers $\mathcal{O}_K$. Let $M_K$ be the set of places (i.e. equivalence classes of multiplicative valuations) on $K$. A place $v$ is called finite if $v$ contains only non-archimedean valuations, and infinite otherwise. $K$ has only finitely many infinite places. Let $S$ be a finite subset of $M_K$, containing all infinite places. A number $a \in K$ is called an $S$-unit if $|a|_v = 1$ for every valuation $| |_v$ from a place $v \in M_K \setminus S$. The $S$-units form a multiplicative group which is denoted by $U_S$. We shall deal with the $S$-unit equation

\[ \alpha_1 x + \alpha_2 y = \alpha_3 \quad \text{in } x, y \in U_S, \tag{1} \]

where $\alpha_1, \alpha_2, \alpha_3 \in K^* (= K \setminus \{0\})$. Lang [9] proved that (1) has only finitely many solutions. Denote this number of solutions by $v_S(\alpha_1, \alpha_2, \alpha_3)$. We call two triples $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ in $(K^*)^3$ (and their corresponding $S$-unit equations) $S$-equivalent if there exist a permutation $\sigma$ of $(1, 2, 3)$, a $\lambda \in K^*$ and $S$-units $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that

\[ \beta_1 = \lambda \varepsilon_{\sigma(1)} \quad \text{for } i = 1, 2, 3. \tag{2} \]

It is easy to check that if $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ are $S$-equivalent, then $v_S(\alpha_1, \alpha_2, \alpha_3) = v_S(\beta_1, \beta_2, \beta_3)$ (cf. [6] § 1).

Evertse [3] proved that $v_S(\alpha_1, \alpha_2, \alpha_3) \leq 3 \times 7^{d+2s}$ for every $(\alpha_1, \alpha_2, \alpha_3) \in (K^*)^3$ where $s$ denotes the cardinality of $S$. A general upper bound for $v_S$ which is polynomial in $s$ does not exist, since a result of Erdös, Stewart and Tijdeman

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implies that in case \( K = \mathbb{Q} \) there is a positive constant \( C \) and there are sets \( S \) of arbitrarily large cardinality for which \( v_S(1, 1, 1) > \exp(C(s/\log s)^{1/2}) \).

On the other hand, for a large class of triples \((\alpha_1, \alpha_2, \alpha_3)\) specified below, Györy [7] derived an upper bound for \( v_S(\alpha_1, \alpha_2, \alpha_3) \) which is linear in \( s \). Let \( p_1, \ldots, p_t \) be the prime ideals corresponding to the finite places in \( S \). For any \( \alpha \in K^* \) the principal ideal \((\alpha)\) can be written uniquely as a product of two (not necessarily principal) ideals \( \alpha' \) and \( \alpha'' \), where \( \alpha' \) is composed of \( p_1, \ldots, p_t \), and \( \alpha'' \) is composed solely of prime ideals different from \( p_1, \ldots, p_t \). We put \( N_S(\alpha) = N_{K/\mathbb{Q}}(\alpha'') \). Györy proved the following.

For any \( \varepsilon \) with \( 0 < \varepsilon \leq 1 \) there is an effectively computable number \( C \) depending only on \( \varepsilon, K \) and \( S \) such that \( v_S(\alpha_1, \alpha_2, \alpha_3) \leq s + 3t \) for each triple \((\alpha_1, \alpha_2, \alpha_3) \in (\mathcal{O}_K \setminus \{0\})^3 \) with

\[
N_S(\alpha_3) \geq C \quad \text{and} \quad (N_S(\alpha_3))^{1-\varepsilon} \geq \min(N_S(\alpha_1), N_S(\alpha_2)).
\] (3)

If, moreover, \((\log N_S(\alpha_3))^{1-\varepsilon} \geq \max(\log N_S(\alpha_1), \log N_S(\alpha_2))\), then \( v_S(\alpha_1, \alpha_2, \alpha_3) \leq s + t \).

We remark that there are infinitely many \( S \)-equivalence classes which have a representative satisfying condition (3) and infinitely many \( S \)-equivalence classes which do not have such a representative (cf. [6], § 3).

In this paper we prove that almost all equivalence classes of \( S \)-unit equations in two unknowns have remarkably few solutions.

**Theorem 1.** Let \( S \) be a finite subset of \( M_K \) containing all infinite places. Then there exists a finite set \( \mathcal{A} \) of triples in \((K^*)^3 \) with the following property: for each triple \((\alpha_1, \alpha_2, \alpha_3) \in (K^*)^3 \) which is not \( S \)-equivalent to any of the triples from \( \mathcal{A} \), the number of solutions of (1) is at most two.

For \( s > 1 \), the upper bound 'two' cannot be improved, since there are infinitely many \( S \)-equivalence classes of \( S \)-unit equations (1) with two solutions (cf. [6], § 1). The proof of Theorem 1 is based on the Main Theorem on \( S \)-Unit Equations (Lemma 1) which is proved by the \( p \)-adic analogue of the Thue-Siegel-Roth-Schmidt method and is therefore ineffective. Consequently, its proof does not enable one to describe triples \((\alpha_1, \alpha_2, \alpha_3)\) for which (1) has no more than two solutions. The following improvement of Györy's result is based on the effective method of Baker and its \( p \)-adic analogue. It provides the upper bound \( s + 1 \) for the number of solutions of all \( S \)-unit equations with the exception of a finite set of \( S \)-equivalence classes which is, at least in principle, effectively determinable. For any non-zero algebraic number \( \alpha \) with minimal polynomial

\[
F(X) = a_0 \prod_{i=1}^{n} (X - \alpha_i) \in \mathbb{Z}[X],
\] (4)

we define the height \( h(\alpha) \) of \( \alpha \) by

\[
h(\alpha) = \left( |a_0| \prod_{i=1}^{n} \max(1, |\alpha_i|) \right)^{1/n}.
\] (5)

For given \( C \geq 1 \), there are only finitely many \( \alpha \in K^* \) with \( h(\alpha) \leq C \), and all these \( \alpha \) can be effectively determined.
Theorem 2. Let $S$ be a finite subset of $M_K$ of cardinality $s$, containing all infinite places. Suppose that the rational primes corresponding to the finite places in $S$ do not exceed $P(\geq 2)$. Let $\mathcal{B}$ denote the set of triples $(\beta_1, \beta_2, \beta_3) \in (\mathcal{O}_K \setminus \{0\})^3$ with

$$\max(h(\beta_1), h(\beta_2), h(\beta_3)) \leq \exp\{(C_1 s)^{C_2 s} P^{d+1}\},$$

where $C_1$ and $C_2$ are certain explicitly computed numbers depending only on $d$ and $|D_K|$. Then for each triple $(\alpha_1, \alpha_2, \alpha_3) \in (K^*)^3$ which is not $S$-equivalent to any of the triples in $\mathcal{B}$, the number of solutions of (1) is at most $s+1$.

For $t > 0$, Theorem 2 implies Győry's result stated above. For let $(\alpha_1, \alpha_2, \alpha_3) \in (\mathcal{O}_K \setminus \{0\})^3$ be a triple satisfying (3) for some $t > 0$ and some number $C$ which will be chosen later. For any triple $(\beta_1, \beta_2, \beta_3) \in (\mathcal{O}_K \setminus \{0\})^3$ which is $S$-equivalent to $(\alpha_1, \alpha_2, \alpha_3)$ we have

$$\{\max(h(\beta_1), h(\beta_2), h(\beta_3))\}^d \geq \frac{N_S(\alpha_3)}{\min(N_S(\alpha_1), N_S(\alpha_2))}, \quad (6)$$

This can be proved easily by observing that the right hand side of (6) does not change if $\alpha_1, \alpha_2, \alpha_3$ are multiplied by the same number in $K^*$ or by different $S$-units, that the left-hand side of (6) is invariant under permutations of $\beta_1, \beta_2, \beta_3$, and that for each $\beta$ in $\mathcal{O}_K \setminus \{0\}$

$$1 \leq N_S(\beta) \leq |N_K(\beta)| \leq (h(\beta))^d.$$

By combining (6) with (3) we obtain that

$$\max(h(\beta_1), h(\beta_2), h(\beta_3)) \leq C^{t/d}$$

for each triple $(\beta_1, \beta_2, \beta_3) \in (\mathcal{O}_K \setminus \{0\})^3$ which is $S$-equivalent to $(\alpha_1, \alpha_2, \alpha_3)$. Together with Theorem 2 this implies that (1) has at most $s+1$ solutions if $C$ is sufficiently large.

By combining Theorem 2 with an explicit upper bound for the heights of the solutions of (1), derived by Győry [7] (see also Lemma 7 in this paper) we obtain that any triple $(\beta_1, \beta_2, \beta_3) \in (K^*)^3$ for which $\beta_1 x' + \beta_2 y' = \beta_3$ has more than $s+1$ solutions in $S$-units $x', y'$, is $S$-equivalent to a triple $(\alpha_1, \alpha_2, \alpha_3) \in (\mathcal{O}_K \setminus \{0\})^3$ such that the solutions of (1) have heights which do not exceed an effectively computable number independent of $\alpha_1, \alpha_2, \alpha_3$. More precisely we have the following result.

Theorem 3. Let $K, S, s, P$ have the same meaning as in Theorem 2. Let $(\beta_1, \beta_2, \beta_3) \in (K^*)^3$ be a triple for which the equation $\beta_1 x' + \beta_2 y' = \beta_3$ in $S$-units $x', y'$ has at least $s+2$ solutions. Then there is a triple $(\alpha_1, \alpha_2, \alpha_3) \in (\mathcal{O}_K \setminus \{0\})^3$, $S$-equivalent to $(\beta_1, \beta_2, \beta_3)$, such that all solutions $(x, y)$ of (1) satisfy

$$\max(h(x), h(y)) \leq \exp\{(C_3 s)^{C_4 s} P^{2d+2}\},$$

where $C_3$ and $C_4$ are effectively computable numbers depending only on $d$ and $|D_K|$.
The special case $K=\mathbb{Q}$ of Theorem 1 has been considered in [6] §5. On the other hand, it is possible to generalize Theorem 1 to the case that $K$ is any subfield of $\mathbb{C}$ and $U_S$ is any finitely generated multiplicative subgroup of $\mathbb{C}^*$, or that $U_S$ is just a subgroup of finite rank of $\mathbb{C}^*$. For the proofs it suffices to replace the Main Theorem on $S$-Unit Equations as we use it (Lemma 1) by the version due to van der Poorten and Schlickewei [14] in the first instance and the version of Laurent [12] in the second.

Suppose that we want to extend our results to $S$-unit equations

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = \alpha_{n+1}$$

in $x_1, \ldots, x_n \in U_S$, \hspace{1cm} (7)

where $(\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbb{K}^*)^{n+1}$ with $n > 2$. If $U_S$ is infinite an equation of this type may have infinitely many solutions such that some non-empty proper subsum of $\alpha_1 x_1 + \ldots + \alpha_n x_n$ vanishes. Such solutions will be called degenerate. For example, let $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{K}^*$ such that $\alpha_1 x_1' + \ldots + \alpha_{n-1} x'_{n-1} = 0$ for some $x_1', \ldots, x'_{n-1} \in U_S$. Then, for any $\varepsilon \in U_S$, Eq. (7) with $\alpha_{n+1} = \alpha_n$ has the degenerate solution $x_1 = \varepsilon x_1', x_2 = \varepsilon x_2', \ldots, x_{n-1} = \varepsilon x_{n-1}', x_n = 1$. However, as we shall show in §5, the number of non-degenerate solutions can also be large. We shall prove that for $K = \mathbb{Q}$ and for any sufficiently large integer $s$ there is a set $S$ of cardinality $s$ and infinitely many $S$-inequivalent $n+1$-tuples $(\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbb{Q}^*)^{n+1}$ for which the number of non-degenerate solutions of the $S$-unit Equation (7) is at least $\exp((4+o(1)) (s/\log s)^{1/2})$ as $s \to \infty$. Thus the constant two in Theorem 1 and the number $s+1$ in Theorem 2 must be replaced by a number at least as large as $\exp((4+o(1)) (s/\log s)^{1/2})$ as $s \to \infty$. On the other hand, recently Evertse and Györy [5] have shown that apart from finitely many $S$-inequivalent $n+1$-tuples $(\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbb{K}^*)^{n+1}$, the solutions of (7) are contained in at most $2^{n+1}$ proper linear subspaces of $K^n$. For $n=2$, this gives a weaker version of our Theorem 1 with the upper bound $2^6$ instead of 2.

For more background material and applications of results on $S$-unit equations, we refer the reader to our survey paper [6] in the Proceedings of the L.M.S. Conference on Transcendence Theory at Durham, England. At this conference, held in July, 1986, Theorem 1 was established.

§ 1. Proof of Theorem 1

Let $n$ be an integer with $n \geq 1$. Points in the vector space $K^{n+1}$ are denoted by $X = (X_0, X_1, \ldots, X_n)$. If we identify pairwise linearly dependent non-zero points in $K^{n+1}$, we obtain the $n$-dimensional projective space $\mathbb{P}^n(K)$. Points in $\mathbb{P}^n(K)$, so-called projective points, are denoted by $X = (X_0 : X_1 : \ldots : X_n)$, where the homogeneous coordinates are in $K$ and are determined up to a multiplicative constant in $K$. We denote the subset of $\mathbb{P}^n(K)$ of projective points with all the homogeneous coordinates in $U_S$ by $\mathbb{P}^n(U_S)$. We shall apply the Main Theorem on $S$-Unit Equations which was first stated by van der Poorten and Schlickewei [14]. Evertse formulated his version of this theorem in terms of $(c, d, S)$-admissible points. Since $\mathbb{P}^n(U_S)$ consists precisely of all $(1, 0, S)$-admissible points, we may use the following statement.
Lemma 1. (Evertse, [4, Theorem 1]). There are only finitely many projective points \( X = (X_0 : X_1 : \ldots : X_n) \in \mathbb{P}^n(U_5) \) such that

\[ X_0 + X_1 + \cdots + X_n = 0 \]  

(8)

with

\[ X_{i_0} + X_{i_1} + \cdots + X_{i_m} = 0 \]

for each proper, non-empty subset \( \{i_0, i_1, \ldots, i_m\} \) of \( \{0, 1, \ldots, n\} \).

Proof of Theorem 1. Since \( (\alpha_1, \alpha_2, \alpha_3) \) and \( (\alpha_1/\alpha_3, \alpha_2/\alpha_3, 1) \) are \( S \)-equivalent, we may assume without loss of generality that \( \alpha_3 = 1 \). Suppose

\[ \alpha_1 x + \alpha_2 y = 1 \]  

(9)

has three distinct solutions \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) in \((U_5)^2\). Then we obtain, after eliminating \( \alpha_1 \) and \( \alpha_2 \),

\[ x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 = 0. \]  

(10)

Note that the expression on the left-hand side does not change value if we interchange all \( x \)'s and \( y \)'s or if we permute the subscripts \( \{1, 2, 3\} \) consistently. Furthermore,

\[ x_1 y_3 = x_2 y_1, \quad x_2 y_3 = x_3 y_2, \quad x_3 y_1 = x_1 y_3, \]  

(11)

since the solutions of (9) are distinct. We shall show that there are only finitely many possibilities for \( x_2/x_1 \) and \( y_2/y_1 \). By the preceding considerations it suffices to prove this claim in each of the following cases:

(a) no proper, non-empty subsum of the left-hand side of (10) vanishes,

(b1) \( x_1 y_2 + x_2 y_3 = 0, \quad x_2 y_1 + x_3 y_2 - x_3 y_1 + x_1 y_3 = 0, \)

(b2) \( x_1 y_2 - x_3 y_2 = 0, \quad x_2 y_1 - x_2 y_3 - x_3 y_1 + x_1 y_3 = 0, \)

(c1) \( x_1 y_2 - x_2 y_1 + x_3 y_3 = 0, \quad x_3 y_2 - x_3 y_1 + x_1 y_3 = 0, \)

(c2) \( x_1 y_2 + x_2 y_3 - x_3 y_1 = 0, \quad x_2 y_1 + x_3 y_2 + x_1 y_3 = 0, \)

(c3) \( x_1 y_2 + x_2 y_3 - x_1 y_3 = 0, \quad x_2 y_1 + x_3 y_2 - x_3 y_1 = 0. \)

Case (a). By Lemma 1 there are only finitely many projective points \((x_1, y_1 : x_2, y_1 : x_2, y_2 : x_3, y_3 : x_3, y_1 : x_1, y_3) \in \mathbb{P}^2(U_5)\). Hence there are only finitely many possibilities for \( x_2/x_1 \) and \( y_2/y_1 \).

Case (b1). No subsum of \( x_2 y_1 + x_3 y_2 - x_3 y_1 + x_1 y_3 \) can vanish by (11), \( x_2 \neq x_3, y_1 \neq y_2 \). By Lemma 1 there are only finitely many projective points \((x_1, y_2 : x_2, y_3) \in \mathbb{P}^1(U_5)\) and \((x_2, y_1 : x_3, y_2 : x_3, y_1 : x_1, y_3) \in \mathbb{P}^3(U_5)\). Hence there are only finitely many possibilities for \( x_1 y_2/x_2 y_3, y_2/y_1, x_2 y_1/x_1 y_3 \), whence for \( x_2 y_3/x_1 y_1, x_2 y_1/x_1 y_3 \), whence for \( x_2^2/x_1^2 \), whence for \( x_2/x_1 \).

Case (b2). This is impossible, since \( y_2 \neq 0, x_1 \neq x_3 \).

Case (c1). By Lemma 1 there are only finitely many projective points \((x_1, y_2 : x_2, y_3) \in \mathbb{P}^2(U_5)\) and \((x_2, y_1 : x_3, y_2 : x_3, y_1 : x_1, y_3) \in \mathbb{P}^2(U_5)\). Hence there are only finitely many possibilities for \( y_2/y_1, x_1 y_2/x_2 y_3 \), whence for \( x_2/x_1 \).
Case (c2). By Lemma 1 there are only finitely many projective points 
\((x_1 y_2 : x_2 y_3 : x_3 y_1) \in \mathbb{P}^2(U_0)\) and 
\((x_2 y_1 : x_3 y_2 : x_1 y_3) \in \mathbb{P}^2(U_0)\). Hence there are only 
finitely many possible values for \(x_2 y_3 / x_1 y_2, x_1 y_2 / x_3 y_1, x_3 y_2 / x_2 y_1, x_2 y_1 / x_1 y_3\), 
whence for \(x_3^2 y_1 / x_1^2 y_2, x_1 y_2^2 / x_2 y_1^2\), whence for \(x_2^2 / x_1^3\) and \(y_2^3 / y_1^3\), whence for \(x_2 / x_1\) and \(y_2 / y_1\).

Case (c3). By Lemma 1 there are only finitely many projective points 
\((x_1 y_2 : x_2 y_3 : x_3 y_1) \in \mathbb{P}^2(U_0)\) and 
\((x_2 y_1 : x_3 y_2 : x_1 y_3) \in \mathbb{P}^2(U_0)\). Hence there are only 
finitely many possibilities for \(x_2 / x_1\) and \(y_2 / y_1\).

We conclude that there are indeed only finitely many possibilities for \(x_2 / x_1\) and \(y_2 / y_1\). Since \((x_1, y_1)\) and \((x_2, y_2)\) satisfy (9), we have

\[
\alpha_1 x_1 = \frac{(y_2/y_1) - 1}{y_2/y_1 - x_2/x_1}, \quad \alpha_2 y_1 = \frac{(x_2/x_1) - 1}{x_2/x_1 - y_2/y_1}
\]

Hence there are only finitely many possibilities for \(\alpha_1\) and \(\alpha_2\) up to multiplicative factors from \(U_0\).

Remark. Up to multiplicative factors from \(U_0\), there are only finitely many 
elements of \(K^*\) which can be represented as sums of two \(S\)-units in two essentially 
different ways. This is an immediate consequence of Lemma 1. It means that 
in Theorem 1 ‘two’ can be replaced by ‘one’ when \(\alpha_1 = \alpha_2\) and solutions \((x, y)\) 
and \((y, x)\) are not distinguished.

§ 2. Valuations and heights

Since the algebraic number field \(K\) has degree \(d\), it has \(d\) different 
\(\mathbb{Q}\)-isomorphisms into \(\mathbb{C}\), \(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+r_1}, \sigma_{r+r_1+1}, \ldots, \sigma_{r+r_1+r_2} = \sigma_d\) say, where 
\(\sigma_i\) maps \(K\) into \(\mathbb{R}\) for \(i = 1, \ldots, r_1\), \(\sigma_i\) maps \(K\) into \(\mathbb{C}\) for \(i = r_1 + 1, \ldots, d\) and 
\(\sigma_i + j(\alpha) = \sigma_i + j(\alpha)\) for \(\alpha \in K\) and \(j \in \{1, \ldots, r_2\}\). \(K\) has exactly \(r_1 + r_2\) infinite 
places, and each infinite place \(v\) contains exactly one valuation of the type 
\(|\sigma_{i(v)}(\ ))|^{d_v/d}\) where \(i(v) \in \{1, \ldots, r_1 + r_2\}\). In each infinite place \(v\) we choose the valuation

\[
|\ |_v = |\sigma_{i(v)}(\ ))|^{d_v/d}, \quad \tag{12}
\]

where \(d_v = 1\) if \(1 \leq i(v) \leq r_1\) and \(d_v = 2\) if \(r_1 + 1 \leq i(v) \leq r_1 + r_2\).

For each \(\alpha \in K^*\) we have

\[
(\alpha) = \prod_p p^{\text{ord}_p(\alpha)},
\]

where \((\alpha)\) denotes the ideal generated by \(\alpha\), \(p\) runs through the set of prime 
ideals of \(\mathcal{O}_K\), and the exponents \(\text{ord}_p(\alpha)\) are integers of which at most finitely 
many are non-zero. If \(v\) is the finite place corresponding to the prime ideal 
\(p\), then we put

\[
|\alpha|_v = (N_{K/Q}(p))^{-\text{ord}_p(\alpha)/d} \quad \text{if} \quad \alpha \neq 0, \quad |0|_v = 0. \quad \tag{13}
\]

The valuations \(|\ |_v (v \in M_\alpha)\) chosen above satisfy the product formula
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\[ \prod_{\nu \in M_K} |\alpha|_{\nu} = 1 \quad \text{for} \quad \alpha \in K^*. \]  

(14)

The set of infinite places on \( K \) is denoted by \( S_\infty \). If \( S \) is any finite subset of \( M_K \) containing \( S_\infty \), then we have

\[ N_S(\alpha) = (\prod_{\nu \in S} |\alpha|_{\nu})^d \quad \text{for} \quad \alpha \in K, \]  

(15)

where \( N_S(\alpha) \) has the same meaning as in the Introduction. In particular, \( N_{S_\infty}(\alpha) = |N_{K/Q}(\alpha)| \). Finally we have

\[ N_S(\alpha) = 1 \quad \text{for each S-unit } \alpha. \]  

(16)

If \( h \) is the height defined in (5), then (cf. [11], p. 54)

\[ h(\alpha) = \prod_{\nu \in M_K} \max(1, |\alpha|_{\nu}) \quad \text{for} \quad \alpha \in K^*. \]  

(17)

We shall use frequently that

\[ h(\alpha^{-1}) = h(\alpha), \quad h(\alpha \beta) \geq h(\alpha) \cdot h(\beta) \quad \text{for} \quad \alpha, \beta \in K^*. \]  

(18)

In the literature two other heights frequently appear, namely \( H(\alpha) \), which is the maximum of the absolute values of the coefficients of the minimal polynomial of \( \alpha \) over \( \mathbb{Z} \), and \( |\alpha|_{\nu} \), which is the maximum of the absolute values of the conjugates of \( \alpha \) over \( \mathbb{Q} \). We have

\[ |\alpha|^{1/n} \leq H(\alpha) \leq |\alpha| \]  

(19)

if \( \alpha \) is a non-zero algebraic integer of degree \( n \), and

\[ \frac{1}{2} H(\alpha)^{1/n} \leq h(\alpha) \leq (n+1)^{1/(2n)} H(\alpha)^{1/n} \]  

(20)

if \( \alpha \) is a non-zero algebraic number of degree \( n \). (19) is obvious, while the proof of (20) can be found, for instance, in [11], p. 60, Theorem 2.8.

§ 3. Lemmas for the proofs of Theorems 2 and 3

We shall use the same notation as in the previous sections. In particular, \( s \) is the cardinality of \( S \) and the rational primes corresponding to the finite places in \( S \) do not exceed \( P(\geq 2) \). Let \( t \) denote the number of finite places in \( S \), and define \( r \) such that \( r+1 \) is equal to the number of infinite places on \( K \). Thus \( s = r + t + 1 \). It is well known that the group \( U_S \) of \( S \)-units has rank \( r + t = s - 1 \). In the remainder of the paper, \( c_1, c_2, \ldots \), will denote effectively computable numbers \( > 1 \), which depend only on \( d \) and the absolute value of the discriminant \( D_K \) of \( K \). We shall use frequently the fact that the class number \( h_K \) of \( K \) and the regulator \( R_K \) of \( K \) can be estimated from above by effectively computable numbers depending only on \( d \) and \( |K| \). This follows from an upper bound
for \( h_K R_K \) derived by Siegel [16] and a lower bound for \( R_K \) due to Zimmert [17].

In the next three lemmas some estimates for \( S \)-units are given. We recall that \( d_v \) and the valuations \( | \cdot |_v \) were introduced in (12) and (13).

**Lemma 2.** If \( r \geq 1 \), then there exist multiplicatively independent units \( \eta_1, \ldots, \eta_r \) in \( \mathcal{O}_K \) with the following properties:

(i) \( \max_j h(\eta_j) \leq c_1 \);

(ii) every unit \( \eta \) in \( \mathcal{O}_K \) can be written as \( \eta = \eta' \eta_1^{a_1} \cdots \eta_r^{a_r} \) with \( a_1, \ldots, a_r \in \mathbb{Z} \) and \( h(\eta') \leq c_2 \);

(iii) for each \( v_0 \in S_\infty \), the entries of the inverse of the matrix
\[
\begin{pmatrix}
|\eta_1|_{v_0} & \cdots & |\eta_r|_{v_0}
\end{pmatrix}
\]
have absolute values at most \( c_3 \).

**Proof.** Lemma 2 has been proved e.g. in [8] Lemma 2 and in [15] Corollaries A.4 and A.5, however with \( |\eta_j| \) and \( |\eta_1| \) instead of \( h(\eta_j), h(\eta') \), respectively. In view of (19) we may replace \( |\eta_j|, |\eta_1| \) by \( h(\eta_j) \) and \( h(\eta') \), respectively.

Let \( \eta_1, \ldots, \eta_r \) be a free system of independent units in \( \mathcal{O}_K \) with the properties specified in Lemma 2, and denote by \( U \) the multiplicative group generated by them.

**Lemma 3.** Let \( \alpha \in K^* \) with \( |N_{K/Q}(\alpha)| = M \). Then there exists an \( \eta \in U \) such that
\[
c_4^{-1} M_d^{d_1/2} \leq |\eta \alpha|_v \leq c_5 M_d^{d_1/2} \quad \text{for every } v \in S_\infty.
\]

**Proof.** This follows e.g. from [8], Lemma 3 or [15], Lemma A.15, together with (12).

Let \( p_1, \ldots, p_t \) be the prime ideals corresponding to the finite places in \( S \). Each of these prime ideals has norm at most \( P_d \). Together with Lemma 3 this proves that there are \( \pi_1, \ldots, \pi_t \in \mathcal{O}_K \) with
\[
(\pi_j) = p_j^{h_j} \quad \text{and} \quad h(\pi_j) \leq c_0 P_{h_K} \quad \text{for } j = 1, \ldots, t. \tag{21}
\]

We fix elements \( \pi_1, \ldots, \pi_t \) in \( \mathcal{O}_K \) with property (21). The number \( \alpha \in K \) is called an \( S \)-integer if \( |\alpha|_v \leq 1 \) for all \( v \in S \) (i.e. \( v \in M_K \setminus S \)). The \( S \)-integers form a ring which is denoted by \( \mathcal{O}_S \). The group of units of \( \mathcal{O}_S \) is just \( U_S \). The next lemma is a straightforward consequence of Lemmas 2 and 3.

**Lemma 4.** Every \( \alpha \in \mathcal{O}_S \) can be written in the form
\[
\alpha = \alpha' \eta_1^{a_1} \cdots \eta_r^{a_r} \pi_1 b_1 \cdots \pi_t b_t \tag{22}
\]
with appropriate rational integers \( a_j, b_j \) and with \( \alpha' \in \mathcal{O}_K \) such that \( \pi_j \nmid \alpha' \) for \( j = 1, \ldots, t \) and
\[
c_7^{-1} N_\mathcal{O}(\alpha)^{d_1/2} \leq |\alpha'|_v \leq c_8 P_d^{d_1/2} N_\mathcal{O}(\alpha)^{d_1/2} \quad \text{for } v \in S_\infty. \tag{23}
\]

**Remark.** It is clear that in (22) \( \alpha/\alpha' \in U_S \).
Proof. Let $\alpha \in \mathcal{O}_S$. Then $(\alpha) = \alpha'' p_1^{d_1} \cdots p_t^{d_t}$, where $\alpha''$ is an integral ideal relatively prime with $p_1, \ldots, p_t$ and $d_1, \ldots, d_t$ are rational integers. Define rational integers $b_j, b_j' (j = 1, \ldots, t)$ by $d_j = h_K b_j + b_j'$ and $0 \leq b_j' < h_K$. Then the ideal $b = \alpha'' p_1^{b_1} \cdots p_t^{b_t}$ is principal, with norm $M$, say. Using the fact that $N_{K/Q}(\alpha'') = N_S(\alpha)$ and that each prime ideal $p_j$ has norm at most $P^{d_j}$, it follows that

$$N_S(\alpha) \leq M \leq P^{\text{id}_K} N_S(\alpha).$$

Together with Lemma 3 and Lemma 2 (ii) this shows that $b$ has a generator $\alpha'$ for which $\pi_j \alpha'$ for $j = 1, \ldots, t$ and (22) and (23) hold. □

We recall that two triples $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)$ in $(K^*)^3$ are called $S$-equivalent if there are $\lambda \in K^*, S$-units $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and a permutation $\sigma$ of $(1, 2, 3)$ such that

$$\beta_i = \lambda \varepsilon_i \alpha_{\sigma(i)} \quad \text{for } i = 1, 2, 3.$$

The next lemma shows that each $S$-equivalence class contains a triple with certain specified properties.

Lemma 5. Each $S$-equivalence class contains a triple $(\alpha_1, \alpha_2, \alpha_3)$ with the following properties:

(i) $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{O}_K \setminus \{0\}$;
(ii) $N_S(\alpha_1) \leq N_S(\alpha_2) \leq N_S(\alpha_3)$;
(iii) $\prod_{v \notin S} \max(|\alpha_1|_v, |\alpha_2|_v, |\alpha_3|_v) \geq c_g^{-1}$;
(iv) $c_7^{-1} N_S(\alpha_i) d_{\alpha_i}^{d_\alpha_i} \leq |\alpha_i|_v \leq c_8 P^{d_\alpha_i h_K} N_S(\alpha_i) d_{\alpha_i}^{d_\alpha_i} d_\alpha_i$ for $i = 1, 2, 3$ and $v \in S_\infty$.
(v) $P^{-h_K} \leq |\alpha_i|_v \leq 1$ for $i = 1, 2, 3$ and $v \in S \setminus S_\infty$.

We shall call such triples $S$-normalized.

Proof. Let $(\beta_1, \beta_2, \beta_3) \in (K^*)^3$. We shall prove that $(\beta_1, \beta_2, \beta_3)$ is $S$-equivalent with an $S$-normalized triple. We suppose that $N_S(\beta_1) \leq N_S(\beta_2) \leq N_S(\beta_3)$. This can be achieved by permuting $\beta_1, \beta_2$ and $\beta_3$. Let $b$ be the inverse of the ideal generated by $\beta_1, \beta_2,$ and $\beta_3$. Then there exists a $\delta \in b$ with $|N_{K/Q}(\delta)| \leq |D_K|^{1/2} N_{K/Q}(b)$ (cf. [10], p. 119 for a sharper estimate). Put $\beta_i = \delta \beta_i$ for $i = 1, 2, 3$. Then $\beta_i \in \mathcal{O}_K \setminus \{0\}$ for $i = 1, 2, 3$, $N_S(\beta_1) \leq N_S(\beta_2) \leq N_S(\beta_3)$ and

$$N_{K/Q}(\beta_1, \beta_2, \beta_3) = N_{K/Q}(\delta) N_{K/Q}(\beta_1, \beta_2, \beta_3) \leq |D_K|^{1/2} N_{K/Q}(b) N_{K/Q}(\beta_1, \beta_2, \beta_3) = |D_K|^{1/2}. \quad (24)$$

Moreover, by (13),

$$N_{K/Q}(\beta_1, \beta_2, \beta_3) = (\prod_{v \notin S_\infty} \max(|\beta_1|_v, |\beta_2|_v, |\beta_3|_v))^{-d} \leq (\prod_{v \notin S} \max(|\beta_1|_v, |\beta_2|_v, |\beta_3|_v))^{-d}.$$
Together with (24) this implies that
\[
\prod_{v \notin \mathcal{S}} \max(|\beta'_1|_v, |\beta'_2|_v, |\beta'_3|_v) \geq c^*_8^{-1}.
\]
Hence the triple \((\beta'_1, \beta'_2, \beta'_3)\) satisfies (i), (ii), (iii). By Lemma 4 there are \(S\)-units \(e_1, e_2, e_3\) such that for \(\alpha_i = e_i \beta'_i\) we have \(\alpha_i \in \mathfrak{O}_K \setminus \{0\}, \pi_i' \alpha_i\) for \(i = 1, 2, 3, j = 1, \ldots, t,\) and
\[
c^*_8^{-1} N(v)(\beta'_j)^{d_v/d_2} \leq |\alpha_i|_v \leq c_8 P_{d_v, t \mathcal{W}_d} N(v)(\beta'_j)^{d_v/d_2} \text{ for } v \in \mathcal{S}_v. \tag{25}
\]
Thus (i) holds. (v) follows from \(\pi_i' \alpha_i\) and (i), while (25) and \(N(v)(\beta'_j) = N(v)(\alpha_i)\) for \(i = 1, 2, 3\) imply (iv). Since \(e_1, e_2, e_3\) are \(S\)-units, \(\alpha_1, \alpha_2, \alpha_3\) also satisfy (ii) and (iii). \(\Box\)

The main tools in the proofs of Theorems 2 and 3 are lower bounds for linear forms in logarithms, both in the archimedean and the \(p\)-adic case.

**Lemma 6.** Let \(v \in S\). Let \(\gamma_1, \ldots, \gamma_k \in K^*\) with \(h(\gamma_i) \leq A_i (3 \leq A_1 \leq \ldots \leq A_k)\) for \(i = 1, \ldots, k\) and let \(b_1, \ldots, b_k\) be rational integers with \(\max |b_i| \leq B(B \geq 3)\). Put
\[
A = \gamma_1^{b_1} \cdots \gamma_k^{b_k} - 1, \quad \Omega = \prod_{i=1}^{k} \log A_i, \quad \Omega' = \prod_{i=1}^{k-1} \log A_i.
\]
Then either \(A = 0\) or
\[
|A|_v \geq \exp \left\{ -(c_{10}k)^{c_{11}k} \Omega \log \Omega' \log B \right\} \quad \text{if } v \text{ is infinite}
\]
and
\[
|A|_v \geq \exp \left\{ -(c_{12}k)^{c_{13}k} P^d (\log P) \Omega (\log B)^2 \right\} \quad \text{if } v \text{ is finite}.
\]

**Proof.** This follows easily from results of Baker [1] (in case that \(v\) is infinite) and van der Poorten [13] (in case that \(v\) is finite), by taking (20) into consideration. \(\Box\)

The next lemma gives an effective upper bound for the heights of the solutions of the \(S\)-unit Equation (1). It is an easy consequence of a result of Györy [7].

**Lemma 7.** Let \(\alpha_1, \alpha_2, \alpha_3\) be non-zero elements of \(\mathfrak{O}_K\) with \(\max h(\alpha_i) \leq A(A \geq 3)\) and let \(x, y \in U_S\) such that
\[
\alpha_1 x + \alpha_2 y = \alpha_3.
\]
Then \(\max(h(x), h(y)) \leq \exp \{(c_{14}s)^{c_{15}sp^{d+1/2} \log A}\}.
\]

**Proof.** Let \(x_3\) be an \(S\)-unit such that \(xx_3, yx_3\) and \(x_3\) are all algebraic integers and put \(x_1 = xx_3\) and \(x_2 = yx_3\). Then \(\alpha_1 x_1 + \alpha_2 x_2 = \alpha_3 x_3\). By a result of Györy [7] there are \(\kappa \in U_S \cap \mathfrak{O}_K\) and \(\rho_1, \rho_2, \rho_3 \in \mathfrak{O}_K\) such that
\[
x_i = \kappa \rho_i \quad \text{for } i = 1, 2, 3,
\]
On $S$-unit equations in two unknowns

and

$$\max_i |r_i| \leq \exp \{ (c_{16} s)^{c_{19} s} P^d (\log P)^{r_1 + \frac{8}{5}} \log A' \},$$

where $A' = \max(3, |a_1|, |a_2|, |a_3|)$. We may now deduce Lemma 7 from this result by employing (19), the inequalities

$$h(x) = h \left( \frac{x_1}{x_3} \right) = h \left( \frac{\rho_1}{\rho_3} \right) \leq h(\rho_1) h(\rho_3),$$

and

$$h(y) = h \left( \frac{x_2}{x_3} \right) = h \left( \frac{\rho_2}{\rho_3} \right) \leq h(\rho_2) h(\rho_3),$$

which hold in view of (18), and the estimate $(\log P)^{r_1 + \frac{8}{5}} \leq P^{1/2} (c_{18} s)^{c_{19} s}$ which applies for appropriate constants $c_{18}$ and $c_{19}$. \(\Box\)

§ 4. Proofs of Theorems 2 and 3

We shall use the same notation as in the previous sections. In particular, $c_{20}, c_{21}, \ldots$ are explicitly computable numbers, depending on $d$ and $|D_\mathfrak{p}|$ only.

Proof of Theorem 2. Let $(\beta_1, \beta_2, \beta_3) \in (K^*)^3$ be an $S$-normalized triple for which the equation $\beta_1 x + \beta_2 y = \beta_3$ in $S$-units $x, y$ has at least $s + 2$ solutions. Put $m = \max(h(\beta_1), h(\beta_2), h(\beta_3))$. We shall prove that

$$m \leq \exp \{ (c_{20} s)^{c_{21} s} P^{d + 1} \}. \quad (26)$$

This proves Theorem 2, since by Lemma 5, each triple in $(K^*)^3$ is $S$-equivalent to an $S$-normalized triple.

Put $\beta'_1 = \beta_1 / \beta_3$, $\beta'_2 = \beta_2 / \beta_3$. By assumption, the equation

$$\beta'_1 x + \beta'_2 y = 1 \quad \text{in} \quad x, y \in U_S$$

has $s + 2$ different solutions $(x_0, y_0), (x_1, y_1), \ldots, (x_{s+1}, y_{s+1})$, say, ordered such that

$$\prod_{\nu \in S} \max(1, |\beta'_1 x_0|_\nu) \leq \prod_{\nu \in S} \max(1, |\beta'_1 x_1|_\nu) \leq \cdots \leq$$

$$\leq \prod_{\nu \in S} \max(1, |\beta'_1 x_{s+1}|_\nu). \quad (27)$$

First we show that for $i = 1, \ldots, s + 1$, there is a place $w(i)$ in $S$ with

$$|\beta'_1 x_i|_{w(i)} \leq P^{c_{22} / s} m^{-1/(c_{23} s^2)} \quad (28)$$

This estimate will play a key role in our proof. To prove (28) we distinguish two cases: (a) $N_\mathfrak{p}(\beta'_1) \leq m^{-d/4}$ and (b) $N_\mathfrak{p}(\beta'_1) > m^{-d/4}$.

We note that the case (a) can essentially be found in Györy [7]. The new aspect of Theorem 2 and its proof is that we can now prove (28), hence the
theorem, in case (b). Further, we shall obtain a slight improvement of Győry [7] in case (a) by treating infinite and finite places uniformly.

Suppose first that $N_S(\beta_1) \leq m^{-d/4}$. Then, by the fact that $x_i$ is an $S$-unit for $i = 1, \ldots, s+1$, and by (16), (15), we have $\prod_{v \in S} |\beta_1 x_i|_{v(0)} \leq m^{-1/4}$ for $i = 1, \ldots, s+1$.

But this implies at once that for each $i$ in \{1, \ldots, s+1\} there is a $w(i) \in S$ with $|\beta_1 x_i|_{w(i)} \leq m^{-1/(4a)}$.

Now suppose that $N_S(\beta_1) > m^{-d/4}$. Then also $N_S(\beta_2) > m^{-d/4}$, by Lemma 5 (ii). Let $i \geq 1$ and take $v \in M_K \setminus S$. By $\beta_1 x_j + \beta_2 y_j = 1$ for $j = 0, 1, \ldots, s+1$, we have

$$|\beta_1 (x_i - x_0)|_v = |\beta_2 (y_0 - y_j)|_v,$$

whence

$$|\beta_1 (x_i - x_0)|_v \leq \min(|\beta_1|_v, |\beta_2|_v) \quad \text{for } v \in M_K \setminus S.$$

Together with the product formula (14) this implies that

$$\prod_{v \notin S} |\beta_1 (x_i - x_0)|_v \geq A, \quad (29)$$

where

$$A = \left(\prod_{v \notin S} \min(|\beta_1|_v, |\beta_2|_v)\right)^{-1}.$$

By applying the product formula and (15) we obtain

$$A = \left(\prod_{v \notin S} |\beta_1 \beta_2|_v\right)^{-1} \prod_{v \notin S} \max(|\beta_1|_v, |\beta_2|_v) \prod_{v \notin S} \max\left(\frac{|\beta_1|_v}{|\beta_3|_v}, \frac{|\beta_2|_v}{|\beta_3|_v}\right).

Another application of the product formula yields that

$$A = N_S(\beta_1 \beta_2)^{1/d} N_S(\beta_3)^{1/d} \prod_{v \notin S} \max(|\beta_1|_v, |\beta_2|_v). \quad (30)$$

In view of $\beta_1 x_0 + \beta_2 y_0 = \beta_3$ we have $|\beta_3|_v \leq \max(|\beta_1|_v, |\beta_2|_v)$ for $v \in M_K \setminus S$. Hence by Lemma 5 (iii),

$$\prod_{v \notin S} \max(|\beta_1|_v, |\beta_2|_v) = \prod_{v \notin S} \max(|\beta_1|_v, |\beta_2|_v, |\beta_3|_v) \geq c_9^{-1}. \quad (31)$$

By Lemma 5 (iv), (v) and the fact that $P \geq 2$ and $N_S(\beta_i) \geq 1$ for $i = 1, 2, 3$ we have

$$|\beta_i|_v \geq P^{-c_4} \max(1, |\beta_i|_v) \quad \text{for } i = 1, 2, 3 \text{ and } v \notin S.$$

Therefore, by (15) and Lemma 5 (ii), we have, for $i = 1, 2, 3$,

$$N_S(\beta_3) \geq N_S(\beta_i) \geq P^{-c_4ds} \left(\prod_{v \notin S} \max(1, |\beta|_v)\right)^d = P^{-c_4ds} \{h(\beta_i)\}^d.$$

Hence

$$N_S(\beta_3) \geq P^{-c_4ds} m^d.$$
Together with (15), (30) and (31) and \( N_\varphi(\beta'_2) \geq N_\varphi(\beta'_1) \geq m^{-d/4} \) this yields
\[
A \geq P^{-c_{25}s}m^{1/2}.
\]

By combining this with (29) we obtain
\[
\prod_{v \in S} |\beta'_1(x_i - x_0)_v|_v \geq P^{-c_{25}s}m^{1/2} \quad \text{for } i = 1, \ldots, s + 1.
\] (32)

Using
\[
|\beta'_1(x_i - x_0)_v|_v \leq 2 \max(1, |\beta'_1 x_0|_v) \max(1, |\beta'_1 x_i|_v) \quad \text{for } v \in S,
\]
we obtain, in view of (27),
\[
\prod_{v \in S} |\beta'_1(x_i - x_0)_v|_v \leq 2^s \left( \prod_{v \in S} \max(1, |\beta'_1 x_0|_v) \right)^2 \left( \prod_{v \in S} \max(1, |\beta'_1 x_i|_v) \right)^2.
\]

Together with (32) this yields
\[
\prod_{v \in S} \max(1, |\beta'_1 x_i|_v) \geq P^{-c_{26}s}m^{1/4}.
\] (33)

We may assume that
\[
m > P^{4c_{26}s},
\] (34)

since otherwise (26) holds for appropriate \( c_{20}, c_{21} \). Now (33) implies that there is a \( v(i) \in S \) with
\[
|\beta'_1 x_i|_{v(i)} \geq P^{-c_{26}m^{1/4(s)}}.
\]

Further, since \( \prod_{v \in S} |\beta'_1 x_i|_v \geq 1 \), there is a \( w(i) \in S \) with
\[
|\beta'_1 x_i|_{w(i)} \leq P^{c_{26}s}m^{-1/(4s^2)}.
\]

This implies (28) for sufficiently large \( c_{22} \).

By using (28) we now prove that for appropriate \( i, j (i \neq j) \) and \( w, |1 - y_i/y_j|_w \) is quite small in terms of \( m \). Then, a standard application of Baker's inequality and its \( p \)-adic analogue will yield a lower bound for \( |1 - y_i/y_j|_w \) in terms of \( m \) which immediately provides inequality (26).

By the box principle, there are distinct \( i, j \in \{1, \ldots, s + 1\} \) with \( w(i) = w(j) = w \), say. Hence
\[
|\beta'_1 x_i|_w \leq P^{c_{22}s}m^{-1/(c_{25}s^2)}, \quad |\beta'_1 x_j|_w \leq P^{c_{22}s}m^{-1/(c_{25}s^2)}.
\] (35)

While proving (26) we assume that
\[
m \geq (4s^2 P^{c_{22}s})^{2c_{21}}
\] (36)
which is obviously no restriction. Then \(|\beta_1' x_i|_w \leq \frac{1}{2}\) and \(|\beta_1' x_j|_w \leq \frac{1}{2}\). Together with (35) and \(\beta_1' x_i + \beta_2' y_i = \beta_1' x_j + \beta_2' y_j = 1\) this shows that

\[
|\beta_2' y_j|_w \leq \frac{1}{2}, \quad |\beta_2' (y_i - y_j)|_w = |\beta_1' (x_j - x_i)|_w \leq 2 P^{c_{22}s} m^{-1/(c_{22}s^2)}.
\]

By combining this with (36) we obtain

\[
\left| 1 - \frac{y_i}{y_j} \right|_w = \frac{|\beta_2' (y_i - y_j)|_w}{|\beta_2' y_j|_w} \leq m^{-1/(2c_{23}s^2)} \leq m^{-1/(c_{21}s^2)}.
\]

(37)

Further, \(y_i/y_j \neq 1\), since \((x_i, y_i)\) and \((x_j, y_j)\) are distinct solutions. By Lemma 4 and \(y_i/y_j \in U_S\), there are rational integers \(a_1, \ldots, a_r, b_1, \ldots, b_t\) such that

\[
\frac{y_i}{y_j} = z \prod_{k=1}^{r} \eta_k \prod_{l=1}^{t} \pi_l^e,
\]

(38)

where \(\eta_1, \ldots, \eta_r\) satisfy the conditions of Lemma 2, \(\pi_1, \ldots, \pi_t\) satisfy the conditions of (21) and

\[
z \in \mathcal{O}_K, \quad h(z) \leq c_{28} P^{c_{29} s}.
\]

(39)

By combining this with Lemma 6, (38), Lemma 2 (i) and (21) we obtain

\[
\left| 1 - \frac{y_i}{y_j} \right|_w \leq \exp \left\{ - (c_{30} s)^{c_{31} s} P^{d + 1/4} (\log 2 B)^2 \right\},
\]

(40)

where \(B = \max(3, |a_1|, \ldots, |a_r|, |b_1|, \ldots, |b_t|)\).

We shall now estimate \(B\) from above. By (18) and Lemma 7 we have

\[
h \left( \frac{y_i}{y_j} \right) \leq h(y_i) h(y_j) \leq \exp \left\{ (c_{32} s)^{c_{33} s} P^{d + 1/2} \log(4 m) \right\}.
\]

(41)

For \(l = 1, \ldots, t\), let \(u_l\) be the finite place in \(S\) for which \(|\pi_l|_{u_l} < 1\). By (38), the product formula, (17) and (18) we have

\[
2^{b_i/d} \leq \max \left( |\pi_l|_{u_l}^{-b_l} |u_l| \right) = \max \left( \left| \frac{y_i}{y_j} z^{-1} \right|_{u_l}, \left| \frac{y_l}{y_j} z^{-1} \right|_{u_l}^{-1} \right)
\]

\[
= \max \left( \left| \prod_{u \neq u_l} \frac{y_i}{y_j} z \right|_{u}, \left| \frac{y_l}{y_j} z \right|_{u_l} \right)
\]

\[
\leq \prod_{u \notin \mathcal{M}_K} \max \left( 1, \left| \frac{y_i}{y_j} z \right|_u \right) = h \left( \frac{y_i}{y_j} z \right) \leq h \left( \frac{y_i}{y_j} \right) h(z),
\]

for \(l = 1, \ldots, t\). Put \(B' = \max_{1 \leq l \leq t} |b_l|\). Together with (39) and (41) this yields

\[
B' \leq (c_{34} s)^{c_{35} s} P^{d + 3/4} \log(4 m).
\]

(42)
Note that, by (38) and (18),
\[
h(\eta_1^{x_1} \cdots \eta_r^{x_r}) = h\left(\frac{v_1}{y_j} z^{-1} \prod_{i=1}^r \eta_i^{-b_i}\right) \\
\leq h\left(\frac{v_1}{y_j}\right) h(z) \left(\prod_{i=1}^r h(\eta_i)\right)^{b_i}
\]
Together with (41), (39), (21) and (42) this implies
\[
h(\eta_1^{x_1} \cdots \eta_r^{x_r}) \leq \exp\{s\, c_{36}\, \lambda^{d+1} \log(4m)\}.
\]
By (17) and (18) we have \(h(\alpha) \geq |\alpha|_0\), \(h(\alpha) \geq |\alpha|_{\infty}^{-1}\) for \(\alpha \in \mathcal{O}_K \setminus \{0\}\), \(\nu \in M_K\). Hence
\[
\sum_{i=1}^r a_i \log |\eta_i|_v \leq (c_{36}\, s)^{c_{37} s} P^{d+1} \log(4m) \quad \text{for } v \in S_{\infty}.
\]
Together with Lemma 2 (iii) this yields
\[
\max_{1 \leq k \leq r} |a_k| \leq (c_{38}\, s)^{c_{39} s} P^{d+1} \log(4m).
\]
By combining this with (42) we obtain
\[
2B \leq (c_{40}\, s)^{c_{40} s} P^{d+1} \log(4m).
\]
A substitution of this into (40) yields that
\[
\left|1 - \frac{v_i}{y_j}\right|_{\infty} \leq \exp\{-(c_{42}\, s)^{c_{43} s} P^{d+1} \log(4m)\}.
\]
By comparing this with (37) we obtain
\[
\frac{\log(4m)}{\log\log(4m)} \leq (c_{44}\, s)^{c_{45} s} P^{d + 1/2}.
\]
It is easy to check that this implies (26). 

**Proof of Theorem 3.** Let \((\beta_1, \beta_2, \beta_3) \in (K^*)^3\) and suppose that the equation
\[
\beta_1 x' + \beta_2 y' = \beta_3 \text{ in } x', y' \in U_S\text{ has at least } s + 2 \text{ solutions}.\text{ Then there exists, by}\n\text{Theorem 2, a triple } (\alpha_1, \alpha_2, \alpha_3) \in (\mathcal{O}_K \setminus \{0\})^3, \text{ S-equivalent to } (\beta_1, \beta_2, \beta_3) \text{ such that}
\]
\[
\log \{\max(h(\alpha_1), h(\alpha_2), h(\alpha_3))\} \leq (C_1\, s)^{C_2 s} P^{d+1}
\]
with the \(C_1\) and \(C_2\) specified in Theorem 2. By combining this with Lemma 7, we obtain that each pair of S-units \((x, y)\) with \(\alpha_1 x + \alpha_2 y = \alpha_3\) satisfies
\[
\max(h(x), h(y)) \leq \exp\{(C_3\, s)^{C_4 s} P^{2d+2}\},
\]
where \(C_3\) and \(C_4\) are effectively computable positive numbers depending only on \(d\) and \(|D_K|\).
§ 5. An example of an S-unit equation in more than two variables with many solutions

At the end of the Introduction we mentioned that for the case of unit equations in \( n > 2 \) variables there do not exist such small upper bounds for the numbers of solutions as those of Theorems 1 and 2 for the case \( n = 2 \). In this section we shall prove this claim by showing that for \( K = \mathbb{Q} \) and for any sufficiently large integer \( s \) there is a set \( S \) of cardinality \( s \) and infinitely many pairwise S-inequivalent \( n + 1 \)-tuples \((\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbb{Q}^*)^{n+1}\) for which (7) has at least \( \exp((4 + o(1)) \cdot (s/\log s)^{1/2}) \) non-degenerate solutions as \( s \to \infty \).

To see this, observe that, by Theorem 3 of [2], for \( s \) sufficiently large there is a set \( W \) of \( s - 1 \) prime numbers, and a positive integer \( c \) such that the equation \( x_1 - x_2 = c \) has at least \( \exp((4 + o(1)) \cdot (s/\log s)^{1/2}) \) solutions in positive integers \( x_1 \) and \( x_2 \) all of whose prime factors are from \( W \). Let \( S \) consist of the infinite place together with those places associated with a prime number from \( W \). Next let \( q_1, q_2, \ldots \) be a sequence of prime numbers such that \( q_1 \) is larger than all of the prime numbers in \( W \) and also larger than \( c + n - 3 \) and such that

\[
q_{i+1} > q_i + c + n - 3 \quad \text{for} \quad i = 1, 2, \ldots.
\]

Then, for \( j = 1, 2, \ldots, \) the S-unit equation

\[
x_1 - x_2 + q_j x_3 + x_4 + \ldots + x_n = c + q_j + n - 3
\]

has at least \( \exp((4 + o(1)) \cdot (s/\log s)^{1/2}) \) solutions in S-units, since we may take \( x_3 = \ldots = x_n = 1 \) and choose \( x_1 \) and \( x_2 \) so that \( x_1 - x_2 = c \). Among them at most \( 2n \) solutions are degenerate, since in any vanishing subsum \( x_1 \) does not occur, \( -x_2 \) has to occur and the number of possible values for \( x_2 \) in a vanishing subsum is at most \( 2n \). Observe by (2) that if \((\alpha_1, \ldots, \alpha_{n+1})\) and \((\beta_1, \ldots, \beta_{n+1})\) are S-equivalent \( n + 1 \)-tuples then there is a permutation \( \sigma \) of \( \{1, \ldots, n+1\} \) such that for all pairs \((i, j)\) with \( 1 \leq i \leq n, 1 \leq j \leq n \) we have

\[
\frac{\beta_i}{\beta_j} = \varepsilon_{i,j} \frac{\alpha_{\sigma(i)}}{\alpha_{\sigma(j)}}
\]

with \( \varepsilon_{i,j} \) an S-unit. Let \( k \geq l \geq 1 \). If the \( n + 1 \)-tuples \((\beta_1, \ldots, \beta_{n+1}) = (1, -1, q_k, 1, \ldots, 1, c + q_k + n - 3)\) and \((\alpha_1, \ldots, \alpha_{n+1}) = (1, -1, q_l, 1, \ldots, 1, c + q_l + n - 3)\) are S-equivalent then

\[
q_k = \varepsilon \frac{\alpha_{\sigma(3)}}{\alpha_{\sigma(1)}} \beta_j
\]

(43)

where \( \varepsilon \) is an S-unit, \( \beta_j \in \{1, -1, c + q_k + n - 3\} \) and \( \alpha_{\sigma(3)}, \alpha_{\sigma(1)} \) are from \( \{1, -1, q_l, c + q_l + n - 3\} \). But by construction the \( q_k \)-adic value of the right-hand side of (43) is 1 which is a contradiction. Thus \((1, -1, q_k, 1, \ldots, 1, c + q_k + n - 3)\) and \((1, -1, q_l, 1, \ldots, 1, c + q_l + n - 3)\) are S-inequivalent for \( k \neq l \).

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References


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