I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.
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1 Introduction

The subject of 4-manifold topology has long been known to be somewhat different than that of other dimensions. For instance, one of the most surprising facts about four dimensions is that given a closed topological 4-manifold $M$, there may be infinitely many distinct smooth structures on $M$. However, in dimensions other than 4, there are only ever finitely many possible smooth structures. Much of the research done during the 1980’s and 1990’s focused on obtaining some kind of classification result for 4-manifolds. It was conjectured relatively early on that complex manifolds might form building blocks for such a classification, but this turned out not to be the case. However, another category of manifolds are the symplectic manifolds: those admitting a closed nondegenerate 2-form. Using a type of gluing operation, the symplectic normal sum, Gompf [7] was able to show that one can produce many new symplectic manifolds from old, and so it was thought that these manifolds might be the right candidate for classification theorems. Unfortunately this turned out not to be the case either, but symplectic manifolds remain important class of manifold to study for their own sake.

One might ask if it’s possible to give a purely topological description of these manifolds. Donaldson and Gompf provided a resolution to this question in two companion theorems which roughly state that symplectic manifolds are the ones that admit the structure of a Lefschetz fibration [9]. These objects are essentially a generalization of a fiber bundle, but with finitely many singular fibers. The ultimate goal of this thesis is to give a survey of Lefschetz fibrations, their properties, and some of the basic classification results that exist at the time of writing.

In section 3, we will give a fairly brief review of the material required to work with Lefschetz fibrations. We assume a reasonable familiarity with singular homology/cohomology, but present some important theorems such as Poincaré duality and the Universal Coefficient theorem. We’ll also give a review of basic covering space theory and the Euler characteristic. Last, we move to a more specialized review of 4-manifolds, giving a precise description of the homology of closed orientable 4-manifolds, and presenting the basics of the intersection form.

In section 4, we present the Lefschetz fibration, and describe the topology as best we can. Some of the proof techniques require the relatively advanced machinery of Kirby calculus and handlebody decompositions, and so in these cases we’ll refer the reader to an appropriate reference, as such machinery is beyond the scope of this project. We’ll show that such fibrations satisfy a homotopy exact sequence like fiber bundles, and use this to focus our attention on the relatively nice connected cases.

Last, section 5 will give a survey of some classification results that exist for Lefschetz fibrations. Some of these results are rather technical, but we will include as many proofs
as is reasonable. In particular, we will have a complete classification for the “genus one” Lefschetz fibrations, i.e., when the fibers look like a genus one surface.

2 Conventions

In this document, we’ll assume that all maps are continuous and probably smooth unless otherwise stated. We’ll also use the following notational conventions:

$I$ will denote the unit interval.

$\Sigma_g$ will denote the genus $g$ surface.

In general, $M$ will denote an $m$-manifold, $X$ will denote a 4-manifold (or sometimes an arbitrary topological space).

3 Background Material

For the convenience of the reader, we include a review of some of the necessary prerequisites for the discussion of Lefschetz fibrations. Some of this material is standard; some of it is less so, but the experienced reader can probably skip this section. We will include many standard results for convenience, but few proofs.

3.1 Basics

For our purposes, we will assume that all manifolds are connected. By a closed manifold, we mean a compact manifold with no boundary. We will also assume a reasonable grasp of singular homology and cohomology. If necessary, an excellent reference to learn the subject is [1].

We recall that for any orientable $m$-manifold $M$, we have $H_m(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$, and so make the usual definition of the fundamental class.

Definition 3.1. An orientation of an $m$-manifold $M$ is a choice of generator for $H_m(M, \partial M; \mathbb{Z})$, which we will call the fundamental class of $M$, and denote $[M]$.

The most important tools for dealing with homology and cohomology classes are the cup and cap products, i.e.,

$$\cup : H^i(M; \mathbb{Z}) \times H^j(M; \mathbb{Z}) \to H^{i+j}(M; \mathbb{Z})$$
\[ \circlearrowright : H^i(M; \mathbb{Z}) \times H_j(M; \mathbb{Z}) \to H_{j-i}(M; \mathbb{Z}) \]

We also have the Kronecker product, obtained by evaluating a cohomology class at a homology class, which we will write as:

\[ \langle , \rangle : H^i(M; \mathbb{Z}) \times H_i(M; \mathbb{Z}) \to \mathbb{Z} \]

The formal development of such products is relatively involved, so we will assume that the reader has some familiarity with them. If not, there is an excellent treatment in [1]. We will not have much need to use these products except while stating some more advanced results, but we'll give a short list of properties for completeness.

**Theorem 3.1.** The following properties hold for the cup product whenever they make sense:

1. \( \cup \) is natural, i.e., if \( f : X \to Y \) then \( f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta) \);
2. Let \( 1 \in H^0(X; \mathbb{Z}) \) denote the class of the augmentation cocycle which takes each 0-simplex to \( 1 \in \mathbb{Z} \). Then we have \( \alpha \cup 1 = 1 \cup \alpha = \alpha \).
3. \( \cup \) is associative, i.e., \( \alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma \).
4. \( \alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha \).

Similar properties hold for the cap product.

One of the most important tools for us will be the Poincaré Duality theorem, as we will deal almost exclusively with oriented closed 4-manifolds. In later sections, we will expressly use this theorem to give a concrete description of the homology and cohomology of such manifolds.

**Theorem 3.2** (Poincaré Duality). Suppose \( M \) is a closed orientable manifold of dimension \( m \). Then the map

\[ \cap [M] : H^k(M; \mathbb{Z}) \to H_{m-k}(M; \mathbb{Z}) \]

\[ f \mapsto f \cap [M] \]

is an isomorphism for all \( k \). The inverse of this map will be denoted \( PD : H_k(M; \mathbb{Z}) \to H^{m-k}(M; \mathbb{Z}) \).

In fact the theorem holds in any coefficient ring, but we will only be concerned with coefficients in \( \mathbb{Z} \). For completeness of this section, we’ll state another important theorem which lets us compute the homology of product spaces relatively easily.

**Theorem 3.3** (Geometric Künneth Theorem). There is a natural exact sequence:
\[ 0 \to (H_*(X) \otimes H_*(Y))_n \to H_n(X \times Y) \to (H_*(X) \ast H_*(Y))_{n-1} \to 0 \]

which splits (not naturally). Here, \([(A_*) \ast (B_*)]_n = \bigoplus_{i+j=n} \text{Tor}_1(A_i, B_j). \] The coefficients can be taken in any PID.

We won’t give any treatment of the Tor functor, as its definition is standard and can be found in any standard algebra book, or [1]. As an easy example, we can compute the homology of the torus:

**Example:** Consider the torus \(T^2 = S^1 \times S^1\). Then using the Künneth Theorem, we have:

\[
\begin{align*}
H_0(T^2) &= H_0(S^1) \otimes H_0(S^1) = \mathbb{Z} \otimes \mathbb{Z} \\
&= \mathbb{Z} \\
H_1(T^2) &= [H_0(S^1) \otimes H_1(S^1)] \oplus [H_1(S^1) \otimes H_0(S^1)] \oplus [H_0(S^1) \ast H_0(S^1)] \\
&= [\mathbb{Z} \otimes \mathbb{Z}] \oplus [\mathbb{Z} \otimes \mathbb{Z}] \oplus [\mathbb{Z} + \mathbb{Z}] \\
&= \mathbb{Z} \oplus \mathbb{Z}
\end{align*}
\]

Similarly, we find that \(H_2(T^2) = \mathbb{Z}\), and it is easy to see that this formula also gives \(H_*(T^2) = 0\) for \(n > 2\).

The last result we will have need of is the Universal Coefficient theorem; it has a few forms, but for us this will be sufficient.

**Theorem 3.4** (The Universal Coefficient Theorem). For singular homology/cohomology, the following sequence is split exact:

\[ 0 \to \text{Ext}(H_{n-1}(X, A; \mathbb{Z})) \to H^n(X, A; \mathbb{Z}) \to \text{Hom}(H_n(X, A; \mathbb{Z}), \mathbb{Z}) \to 0 \]

Again, this holds for more general coefficient groups than \(\mathbb{Z}\), but this will be adequate. For a nice proof of this result, the reader can consult [1][p. 282].

### 3.2 The Euler Characteristic

The Euler characteristic is a commonly used numerical invariant one can assign to topological spaces with finite homology.
Definition 3.2. For any topological space $X$ with finite homology, we define the Euler characteristic of $X$, denoted $\chi(X)$ to be:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i b_i$$

where $b_i$ is the $i^{th}$ Betti number of $X$, i.e., the free rank of the $i^{th}$ homology of $X$. By assumption, only finitely many terms are nonzero, so this sum is finite.

Since homology is invariant under homotopy type, so too is the Euler characteristic. For sufficiently nice spaces, we can actually compute the Euler characteristic without the homology.

Proposition 3.1. If $X$ is a finite CW-complex, then:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i k_i$$

where $k_i$ is the number of $i$-cells in any given decomposition. In particular, $\chi(X)$ is independent of the choice of cell decomposition.

Proof. Fix a cell decomposition of $X$, and consider the associated sequence of abelian groups:

$$\cdots \to 0 \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Set $Z_i = \ker \partial_i$ and $B_i = \text{im} \partial_{i+1}$. Then we have the exact sequences from homology:

$$0 \to Z_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

$$0 \to B_i \xrightarrow{\partial_{i+1}} Z_i \xrightarrow{\pi_i} H_i \to 0$$

From the additivity of rank for short exact sequences of abelian groups, we get:

$$k_i = \text{rank}(C_i) = \text{rank}(Z_i) + \text{rank}(B_{i-1})$$

$$\text{rank}(Z_i) = \text{rank}(B_i) + \text{rank}(H_i)$$

and so substituting we obtain:

$$\sum_{i=0}^{\infty} (-1)^i k_i = \sum_{i=0}^{\infty} (-1)^i [\text{rank}(H_i) + \text{rank}(B_i) + \text{rank}(B_{i-1})]$$

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This lets us easily calculate the Euler characteristic of some nice spaces. For instance, one can easily see that 
\[ \chi(S^n) = 1 + (-1)^n, \quad \chi(T^2) = 0, \quad \chi(I) = 1 \] 
by using the obvious cell decompositions. Basic properties of homology show that the Euler characteristic is additive 
under disjoint unions, and one can also show that \( \chi(M \times N) = \chi(M)\chi(N) \). Another easy result that we will use later is the formula for connected sums: for two connected closed 
\( m \)-manifolds \( M \) and \( N \), we have:

\[ \chi(M \# N) = \chi(M) + \chi(N) - \chi(S^m) \]

### 3.3 Fiber Bundles and Fibrations

We recall the definition and basic properties of fiber bundles, and more generally of fibrations.

**Definition 3.3.** A fiber bundle consists of the following topological spaces: \( B \) (the base), \( E \) (the total space), \( F \) (the fibre), and a continuous surjection \( \pi : E \to B \). We require that for all \( x \in E \), there is an open neighborhood \( U \subset B \) of \( \pi(x) \) (a local trivialization) such that there is a homeomorphism \( \phi : \pi^{-1}(U) \to U \times F \), where the following diagram commutes:

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\phi} & U \times F \\
\downarrow{\pi} & & \downarrow{\text{proj}_1} \\
U & & \\
\end{array}
\]

We often assume that the base space is path connected. One can easily see that the fibers above each point must be homeomorphic. One can also define a smooth fiber bundle, in which all maps are smooth, and the fibers above any two points are diffeomorphic.

There is another, more general notion called a fibration. While the definition is fairly abstract, it generalizes the properties of fiber bundles.

**Definition 3.4.** A map \( p : Y \to B \) is called a fibration if it satisfies the homotopy lifting property for all topological spaces \( X \), that is, the following diagram can always be completed:
If it satisfies the homotopy lifting property for all disks $D^n$, it is called a Serre fibration.

**Example:** A projection map $p : B \times F \to B$ is a fibration, as one can easily see that given any maps $f : X \times \{0\} \to B \times F$ and $g : X \times I \to B$, the diagram can be completed with the map $h(x,t) : f(x,t) \times gq(x,0)$, where $q : B \times F \to F$ is the projection in the second coordinate. Hence, any trivializable fiber bundle is a fibration; in fact, all fiber bundles are. The following result shows that fibrations are indeed a generalization of fiber bundles, but the proof is somewhat technical; [1][p. 453] gives a full proof.

**Theorem 3.5.** For a map $p : Y \to B$, the property of being a Serre fibration is a local property in $B$.

**Corollary 3.1.** Any fibre bundle is a fibration.

It is easy to prove that the Euler characteristic multiplies over fiber bundles. In fact, with reasonable assumptions the same is true for fibrations. This result is proven in [14][p. 481]; we will not do so here.

**Theorem 3.6.** Suppose $p : Y \to B$ is a fibration orientable over a field, with fiber $F$ and with base $B$ path connected. Then if $\chi(B), \chi(F)$ are defined, so is $\chi(E)$, and we have $\chi(E) = \chi(B) \cdot \chi(F)$.

One of the most useful properties of fibrations is the induced homotopy exact sequence.

**Theorem 3.7.** If $p : Y \to B$ is a fibration and $y_0 \in Y, b_0 = p(y_0)$, and $F = p^{-1}(b_0)$, then taking $y_0$ as the base point of $Y$ and of $F$ and $b_0$ as the base point of $B$, we have the exact sequence:

$$
\cdots \to \pi_n(F) \xrightarrow{i_*} \pi_n(Y) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial_*} \pi_{n-1}(F) \to \cdots \\
\cdots \to \pi_1(Y) \to \pi_1(B) \to \pi_0(F) \to \pi_0(B)
$$

where $i_*, p_*$ are the induced maps on homotopy group, and $\partial_*$ is the “connecting homomorphism.” While the 0th homotopy is a pointed set rather than a group, the sequence is still exact as maps on sets there.

In the interest of moving forward, we will also leave this proof to the reader. One can also consult [1][p. 453] for a proof.
3.4 Covering Space Theory

Most of this material is standard, but we will have occasional need of some basic results. We’ll summarize the important classical results without proof for the convenience of the reader. For a complete treatment of this subject, one can consult Chapter 3 of Bredon’s excellent textbook [1].

We recall the following definitions:

**Definition 3.5.** A covering space of a topological space $X$ is a space $\tilde{X}$ together with a continuous map $p : \tilde{X} \to X$ such that for each $x \in X$, there is an open neighbourhood $U$ such that each path component of $p^{-1}(U)$ is mapped homeomorphically onto $U$ by $p$.

It is a standard result that the sets $p^{-1}\{x\}$ have the same cardinality for each $x \in X$, and so we refer to this as the number of sheets of the covering space. In the language of the previous section, we see that a covering space is just a fiber bundle with a discrete fiber. The importance of covering spaces comes from results that relate the fundamental group of the base space to that of the cover; for instance we have:

**Theorem 3.8.** Let $(\tilde{X}, p)$ cover $X$, $\tilde{x}_0 \in \tilde{X}$, and $x_0 = p(\tilde{x}_0)$. Then the induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

A cover $\tilde{X}$ of a space $X$ is called a universal cover if $\tilde{X}$ is simply connected.

**Theorem 3.9.** Suppose that $X$ has a universal cover. Then for any subgroup $G$ of $\pi(X)$, there is a covering space $\tilde{X}$ such that $p_*(\pi(\tilde{X})) = G$.

The question of when $X$ has a universal cover has a precise technical resolution giving conditions on $X$, but this won’t be of much consequence. All manifolds, for instance, enjoy this property.

3.5 4-Manifold Specifics

This material is somewhat less standard, so we will give a better exposition.

We’ll begin by giving a concrete description of the homology of orientable closed 4-manifolds, which will be indispensable. If $X$ is a closed orientable manifold, the homology modules are finitely generated, so write $H_i(X; \mathbb{Z}) \cong F_i \oplus T_i$, where $F_i$ is free, and $T_i$ is the torsion submodule. The following lemma is from [12].

**Lemma 3.1.** Let $X$ be a closed oriented 4-manifold. The homology and cohomology mod-
ules are:

\[
\begin{align*}
H_0(X; \mathbb{Z}) &\cong \mathbb{Z} & H^0(X; \mathbb{Z}) &\cong \mathbb{Z} \\
H_1(X; \mathbb{Z}) &\cong F_1 \oplus T_1 & H^1(X; \mathbb{Z}) &\cong F_1 \\
H_2(X; \mathbb{Z}) &\cong F_2 \oplus T_1 & H^2(X; \mathbb{Z}) &\cong F_2 \oplus T_1 \\
H_3(X; \mathbb{Z}) &\cong F_1 & H^3(X; \mathbb{Z}) &\cong F_1 \oplus T_1 \\
H_4(X; \mathbb{Z}) &\cong \mathbb{Z} & H^4(X; \mathbb{Z}) &\cong \mathbb{Z}
\end{align*}
\]

Proof. By Poincaré duality, it suffices to compute about half the list. Since \(X\) is connected, we get \(H_0(X) \cong \mathbb{Z}\), and since \(X\) is closed and oriented, we get \(H_4(X, \partial X) \cong H_4(X) \cong \mathbb{Z}\).

To get the rest, we use the Universal Coefficient theorem; since the sequence splits, we have:

\[
H^i(X) \cong \text{Ext}(H_{i-1}(X), \mathbb{Z}) \oplus \text{Hom}_{\mathbb{Z}}(H_i(X), \mathbb{Z}) \cong F_i \oplus T_{i-1}
\]

The last isomorphism follows from the fact that \(\text{Ext}(H, \mathbb{Z})\) is isomorphic to the torsion submodule of \(H\) if \(H\) is finitely generated, and similarly that \(\text{Hom}(H, \mathbb{Z})\) is isomorphic to the free submodule of \(H\) if \(H\) is finitely generated [10][p. 196]. Now, with Poincaré duality, we have:

\[
\begin{align*}
F_3 \oplus T_2 &\cong H^3(X) \cong H_1(X) \cong F_1 \oplus T_1 \\
F_2 \oplus T_1 &\cong H^2(X) \cong H_2(X) \cong F_2 \oplus T_2 \\
F_1 \oplus T_0 &\cong H^1(X) \cong H_3(X) \cong F_3 \oplus T_3
\end{align*}
\]

But this immediately gives us that \(T_1 \cong T_2, \ T_3 \cong T_0 = 0\) and \(F_1 \cong F_3\), which finishes the proof.

Remark: We note that the only source of torsion in the homology of a 4-manifold originates in \(H_1(X)\); in the special case that \(X\) is simply connected, the homology/cohomology will be particularly easy to calculate.

Now we’ll give a brief review of intersection forms on a 4-manifolds, mostly by following [12]. We begin with some definitions:

Definition 3.6. For a closed oriented 4-manifold \(X\), its intersection form is the symmetric bilinear form:

\[
Q_X : H^2(X; \mathbb{Z}) \times H^2(X; \mathbb{Z}) \to \mathbb{Z}
\]

given by:

\[
Q_X(\alpha, \beta) = \langle \alpha \cup \beta, [X] \rangle
\]

where \([X] \in H_4(X; \mathbb{Z})\) denotes the fundamental class of \(X\).
Note that since both the cup product and the Kronecker product are bilinear, $Q_X$ is also bilinear. Moreover, since $\dim X = 4$, the cup product is symmetric, and so this does define a symmetric bilinear form. One can define the intersection form on manifolds with boundary, as in [9], but for our purposes this is sufficient.

Since $X$ is closed, we can use Poincaré duality to define $Q_X$ on $H_2(X) \times H_2(X)$, by

$$Q_X(\alpha, \beta) = \langle PD(\alpha) \cup PD(\beta), [X] \rangle$$

It is clear that if either $\alpha$ or $\beta$ has torsion, $Q_X(a, b) = 0$. Hence, if $T$ denotes the torsion $\mathbb{Z}$-submodule of $H_2(X)$ (since $X$ is closed, its homology/cohomology is finitely generated so this is possible) we can consider $Q_X$ as just defined on $H_2(X)/T$. Since this is a free $\mathbb{Z}$-module, say of rank $r$, we can choose a basis for $H_2(X)/T$ and represent $Q_X$ as a $r \times r$ matrix with respect to this basis.

Following [8], we now make some appropriate definitions for this situation.

**Definition 3.7.** Suppose $A$ is a free $\mathbb{Z}$-module with symmetric bilinear form $Q$. Let $P$ be the matrix of $Q$ with respect to some basis for $A$. Then:

- The rank of $Q$ is the rank of $A$ as a $\mathbb{Z}$-module. Note that in the case that $A = H_2(X)$, this is exactly the second Betti number of $X$.
- Consider $P$ as a matrix over $\mathbb{R}$ or $\mathbb{Q}$ and diagonalize. Let $b^+_2$ be the number of positive eigenvalues, and $b^-_2$ be the number of negative eigenvalues. The signature of $Q$ is defined as $\sigma(Q) = b^+_2 - b^-_2$.
- If an orientation for $X$ has been chosen, we usually write $\sigma(X)$ instead of $\sigma(Q_X)$.

Since this section deals with intersection forms, one might hope to relate this discussion to the actual intersection of submanifolds. In fact, this is the case. We define the intersection product more generally by:

**Definition 3.8.** Let $M$ be an $m$-manifold. The intersection product is defined:

$$\bullet : H_i(M; \mathbb{Z}) \times H_j(M; \mathbb{Z}) \to H_{i+j-m}(X; \mathbb{Z})$$

by $PD(a \bullet b) = PD(a) \cup PD(b)$.

One can easily check that this satisfies the usual axioms of associativity, and in the case that $M$ is a 4-manifold, commutativity. This product is indeed dual to the intersection form, as we have:

$$Q_X(a, b) = \langle PD(a) \cup PD(b), [X] \rangle = \langle PD(a \bullet b), [X] \rangle = \langle 1, a \bullet b \rangle$$
by definition of the Poincaré dual.

Now we have the following highly useful fact, proved in [1][p. 372].

**Fact 3.1.** Suppose $M$ is an oriented $m$-manifold, and $N, K$ are oriented $n$- and $k$-manifolds, respectively. If $N$ and $K$ intersect transversely, then we have:

$$[K \cap N] = [N] \cdot [K] \in H_{n+k-m}(M; \mathbb{Z})$$

Hence, we have $PD([N \cap K]) = PD([N]) \cup PD([K])$, and so in a sense, the cup product is dual to the transverse intersection of submanifolds.

The signature turns out to be a finer numerical invariant than the Euler characteristic, but it also obeys some nice properties. For instance, it is clear that if $\overline{X}$ denotes $X$ with the opposite orientation, then $\sigma(\overline{X}) = -\sigma(X)$. More importantly, the signature is additive under connected sums [12], i.e.,

$$\sigma(X_1 \# X_2) = \sigma(X_1) + \sigma(X_2)$$

To finish this section, we will compute $\sigma(\mathbb{CP}^2)$, as we will need it later. For a plethora of other calculations, [12][p. 120-126] has an excellent treatment.

**Example:** Indeed, we recall that $\mathbb{CP}^2$ has Betti numbers $b_0 = 1, b_2 = 1, b_4 = 1$, and no other homology. More importantly, $H_2(\mathbb{CP}^2) = \mathbb{Z}[[\mathbb{C}P^1]]$, where $[[\mathbb{C}P^1]]$ denotes the homology class of the projective line [12]. Since any two projective lines intersect in exactly one point, we conclude that $Q_{\mathbb{CP}^2} = [+1]$, i.e., $\sigma(\mathbb{CP}^2) = 1$ [12][p. 124].

**Remark:** We note that this example alone shows the utility of the signature. The above calculation shows that $\mathbb{CP}^2$ cannot be oriented-homotopy equivalent to $\overline{\mathbb{C}P^2}$, while the Euler characteristic only gives $\chi(\mathbb{CP}^2) = \chi(\overline{\mathbb{C}P^2}) = 3$.

4 Lefshetz Fibrations

We now move to the core topic of this thesis: the Lefschetz fibration. These spaces are a kind of fibration that generalizes the idea of a fiber bundle; as we will show, they are essentially a fiber bundle except possibly at a finite number of singular points. As mentioned in the introduction, some of these results will have to be taken on faith at this point, as their proofs lie well beyond the scope of this project. However, a reader familiar with Kirby calculus should be able to fill in any details. As much as possible, we’ll point out exactly which facts are required, and reference a specific resource of [9] to consult.
4.1 Definitions

Definition 4.1. Suppose $X$ is a compact, connected, oriented, smooth 4-manifold. A Lefshetz fibration of $X$ is a map $\pi : X \to B$, where $B$ is a compact connected oriented surface, $\pi^{-1}(\partial B) = \partial X$, and such that each critical point of $\pi$ lies in $\text{Int}X$ and has a local coordinate chart (consistent with the orientation of $X$) of the form $\pi(z_1, z_2) = z_1^2 + z_2^2$. We'll usually denote the set of critical values by $C$, and the set of regular values by $B^*.$

Remark: For our purposes, we will assume that all critical points of $\pi$ lie in distinct fibers of $\pi$. One could perturb the critical points slightly to achieve this, but in general, there is no obvious canonical way. Many authors in the literature simply include this as part of the definition [7], and we will follow this convention.

While it doesn’t look like it, these fibrations do behave much like fiber bundles. To prove this, recall the following theorem due to Ehresmann [5].

Theorem 4.1 (Ehresmann). If $f$ is a smooth mapping between smooth manifolds $M, N$ which is a proper map and a surjective submersion, then $f$ is a fiber bundle.

With it, we can prove:

Proposition 4.1. Let $\pi : X \to B$ be a Lefshetz fibration, and let $C$ be the collection of critical values of $\pi$. Then $\pi := \pi|_{\pi^{-1}(B^*)}$ is a fiber bundle.

Proof. We have:

(1) $\tilde{\pi}$ is a submersion, by definition.

(2) We claim that $\tilde{\pi}$ is a surjection. To see this, note first that since $X$ is compact, there can be only finitely many critical points of $\pi$, since they are isolated. Hence $\Sigma \setminus C$ is still path connected. However, note that $\tilde{\pi}$ is an open map, and so $\text{im} \tilde{\pi}$ is open. However, $X$ is compact, so $X \setminus \pi^{-1}(C)$ is still compact, and so $\text{im} \tilde{\pi}$ is closed. Hence, $\text{im} \tilde{\pi}$ is clopen, and so $\tilde{\pi}$ is surjective.

(3) Lastly, $\pi$ is certainly a proper map, and so $\tilde{\pi}$ must also be proper.

By Ehresmann’s theorem, we conclude that $\tilde{\pi}$ is indeed a fiber bundle.

By Sard’s theorem, the regular fibers are all orientable compact 2-manifolds. If we know that the fibers are connected (which we will prove is the case under reasonable conditions in the next section) then by the classification of compact connected 2-manifolds, the fibers are some genus $g$ surface $\Sigma_g$. We will refer to this as a genus $g$ Lefshetz fibration, and focus our attention almost exclusively on this case.
4.2 The Local Topology of Lefschetz Fibrations

To obtain a better understanding of the topology of a Lefschetz fibration, we’ll try to give a description of the behavior near a singular fiber, following [7]. For the moment, suppose that $\pi : X \to D^2$ is a Lefshetz fibration with just one critical value $p \in D^2$. Then we can find local coordinate charts where $\pi(z_1, z_2) = z_1^2 + z_2^2$, with $\pi(0, 0) = p$. Generically, a nearby fiber has the form $\pi^{-1}(t)$ for some small $t \in \mathbb{R}$ (adjusting coordinate charts if necessary). Hence, the fibers are diffeomorphic to the submanifold of $\mathbb{C}^2$ described by:

$$S = \{x_1^2 + x_2^2 - y_1^2 - y_2^2 = t, x_1y_1 + x_2y_2 = 0; x_i, y_i \in \mathbb{R}\}$$

Intersecting this with a copy of $\mathbb{R}^2$ (i.e., the $x_1, x_2$ plane), we obtain a copy of $S^1$ given by $\{x_1^2 + x_2^2 = t; x_i \in \mathbb{R}\}$, and if we let $t$ approach 0, this circle shrinks to a single point. On the other hand, no such shrinking happens along the other axes. We see that $\pi^{-1}(p)$ can be thought of as obtained from a generic fiber by “pinching” it along some loop. We formally introduce some terminology to describe this situation:

**Definition 4.2.** The curve on $\Sigma_g$ which determines the singular fiber at a point will be called the vanishing cycle for that fiber. If the curve separates $\Sigma_g$ into two surfaces of nonzero genus, it will be called a separating cycle; otherwise it will be called a nonseparating cycle.

![Figure 1: On the left, a separating cycle. On the right, a non-separating cycle.](image)

**Remark:** The vanishing cycle of a singular fiber, say $\pi^{-1}(p)$, actually completely determines the topology of $X$ near $\pi^{-1}(p)$. In fact, any two singular fibers described by non-separating cycles will have diffeomorphic neighbourhoods in $X$, as will two singular fibers described by separating cycles which separate $\Sigma_g$ into two surfaces of the same genus. Unfortunately, the proof of this fact relies on handlebody descriptions, so we will simply take it as given and refer the more experienced reader to [7]. This fact will be convenient for us, as there are relatively few different loops on $\Sigma_g$.

Following the convention of most authors, we will also assume that all Lefschetz fibrations are relatively minimal, that is, no singular fiber contains an embedded sphere of self-intersection $-1$. We note that if a separating cycle separates a fiber $\Sigma_g$ into components of genus $g$ and 0, then it will fail to be relatively minimal. Indeed, let $\Sigma \cup S^2$ be some
singular fiber separated into genus $g$ and $0$ parts, and $\Sigma'$ be a nearby regular fiber of genus $g$. For notational convenience, denote the copy of $S^2$ by $S$.

![Diagram of fibers and $S$](image)

Figure 2: An illustration of $\Sigma'$, $\Sigma$, and $S$ (in genus 2).

Then we have (denoting the intersection product as multiplication):

$$0 = [S] \cdot [\Sigma'] = [S] \cdot ([S] + [\Sigma]) = [S]^2 + [S] \cdot [\Sigma] = [S]^2 + 1$$

by standard intersection theory arguments ($S$ and $\Sigma$ have just one point in common). Thus $[S]^2 = -1$, and so the fibration is not relatively minimal.

With the above fact in hand, it is an easy matter to classify all possible vanishing cycles, and hence distinct kinds of singular fibers up to diffeomorphism. For a genus $g$ Lefschetz fibration, there are $\lfloor g/2 \rfloor$ distinct separating cycles, and $1$ non-separating cycle, for a total of $1 + \lfloor g/2 \rfloor$.

### 4.3 The Homotopy Exact Sequence

We would like to say something about the connectedness of the fibers of a Lefshetz fibration. To do this, we'll prove some results about homotopy exact sequences, similar to the case of regular fibrations.

**Theorem 4.2.** Let $\pi : X \to B$ be a Lefschetz fibration with fiber $F$. Then the maps $F' \hookrightarrow X \to B$ induce an exact sequence:

$$\pi_1(F) \to \pi_1(X) \to \pi_1(B) \to \pi_0(F) \to 0$$

where the map $\pi_1(B) \to \pi_0(F)$ is exact as a function between sets.

Before we prove this, we give the desired corollary:
Corollary 4.1. We can always assume that the fibers of a Lefschetz fibration are connected.

Proof. Suppose that the regular fibers of our Lefschetz fibration are not connected. Then we note that the image of $\pi_1(X)$ must be some finite index subgroup of $\pi_1(B)$ ($F$ is a compact 2-manifold, and hence has only finitely many connected components), and so by standard covering space theory, there is a corresponding covering space $p : \tilde{B} \to B$, with $p_*(\pi_1(\tilde{B})) = \pi_*(\pi_1(X))$. Since $X$ is connected, we can lift $\pi$ to a map $\tilde{\pi} : X \to \tilde{B}$, and view this as our new Lefschetz fibration. Locally, it will act the same, but will have connected fibers [9][p. 290].

Indeed, since this new map is still a Lefschetz fibration, we have an exact sequence (surjectivity at $\pi_1$ follows from the construction of the covering space):

$$\pi_1(\tilde{F}) \to \pi_1(X) \to \pi_1(\tilde{B}) \to \pi_0(\tilde{F}) \to 0$$

By exactness, we conclude that $\pi_0(\tilde{F}) = \{0\}$ and so $\tilde{F}$ is 0-connected, i.e., connected. □

In fact, in most cases we will not even need to do anything. If $B$ is simply connected, i.e., if $B = D^2$ or $B = S^2$, then the exact sequence immediately shows that $\pi_0(F) = \{0\}$, and so no modification is necessary. We will focus all of our attention on this case.

Corollary 4.2. If $\pi : X \to B$ is a Lefschetz fibration, and $B$ is simply connected, then the fibers of $\pi$ are connected.

Now we’ll give a proof of the theorem.

Proof of Theorem 4.2. $\pi : X \to B$ is a fiber bundle away from the singular values, and so as a fibration satisfies the homotopy lifting property. We would like to define a map $\phi : \pi_1(B) \to \pi_0(F)$ in the following way:

Given a loop $\alpha : [0, 1] \to B$ which avoids critical values and any $x \in \pi^{-1}(\alpha(0))$, the homotopy lifting properties ($\pi$ is a fiber bundle there) guarantees a lift $\tilde{\alpha} : [0, 1] \to X$ with $\alpha = \pi \circ \tilde{\alpha}$. We set $\phi(\alpha) = \tilde{\alpha}(1)$, as we would in the case of a fiber bundle. While $\pi$ is not completely a fiber bundle, the situation can still be resolved.

If $\alpha$ avoids the critical values in $B$, we will have no problems. If it does pass through a critical value, say $p \in B$, we will use a local homotopy to adjust $\alpha$ away from $p$. To check that this is well defined, it suffices to check that $\alpha$ can’t change connected components as it passes through $p$. However, we saw in the previous section that $\pi^{-1}(p)$ is obtained from a regular fiber $\Sigma_g$ by collapsing an embedded copy of $S^1$; this will not connect previously disconnected components of the fiber. Hence we conclude that $\phi$ is well defined, and so we do have a map $\phi : \pi_1(B) \to \pi_0(F)$.

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To check exactness at \( \pi_1(B) \), we notice that lifts of a loop in \( \pi_1(B) \) lift to a loop in \( X \) (i.e., are in the kernel of \( \phi \)) if and only if the lift begins and ends in the same component of a fiber.

Similarly, to check exactness at \( \pi_1(X) \), we note that a loop in \( \pi_1(B) \) is constant if and only if its lift is contained in a single fiber.

\[ \square \]

### 4.4 Mapping Class Groups and Dehn Twists

For the time being, we continue to assume that \( \pi : X \to D^2 \) is a Lefschetz fibration with only one singular fiber, say \( \pi^{-1}(p) \). In the definition of a Lefshetz fibration, we required the singular fibers to be in the interior of \( X \), so the boundary \( \partial X \) is actually a \( \Sigma_g \) bundle over \( S^1 \). Hence after we make an identification \( \varphi \) of \( \pi^{-1}(1) \subset X \) with \( \Sigma_g \) (considering \( S^1 \subset \mathbb{C} \)), it takes the form:

\[
\partial X = \frac{\Sigma_g \times I}{(\psi(x),0) \sim (x,1)}
\]

where \( \psi : \Sigma_g \to \Sigma_g \) is a homeomorphism (that depends on our choice of the identification of the reference regular fiber). This map will be called the monodromy of the fibration. It turns out that these ideas give us a nice way to translate questions about Lefschetz fibrations into questions in group theory, which offers a significant advantage. To give a formal treatment of this situation, we’ll take a brief foray into the land of mapping class groups.

**Definition 4.3.** For a smooth manifold \( M \), we define the mapping class group of \( M \), denoted \( \text{Mod}(M) \), to be the group of all isotopy classes of orientation preserving diffeomorphisms of \( M \), i.e.,

\[
\text{Mod}(M) := \frac{\text{Diffeo}^+(M)}{\text{Isotopy}}
\]

For a fiber bundle, we are interested in the kinds of diffeomorphisms one can apply to the fibers.

**Definition 4.4.** The mapping class group of a smooth fiber bundle \( \pi : E \to B \) is defined to be the mapping class group of \( F \).

For a fixed identification \( \varphi \) of a fiber \( F \) over a base point of \( B \), we can define the monodromy representation map \( \Psi : \pi_1(B) \to \text{Mod}(F) \) in the following way. For a loop \( \alpha : I \to B \), the pullback bundle \( \pi_\alpha : \alpha^*(E) \to I \) induces a diffeomorphism \( \psi_\alpha : \pi_\alpha^{-1}(0) \to \pi_\alpha^{-1}(1) \). Using
our reference map \( \varphi \), we thus obtain an element \( \Psi(\alpha) \in \text{Mod}(F) \). It is easy to check that a homotopy of loops in \( B \) will induce an isotopy between diffeomorphisms on \( F \), and so \( \Psi \) is well defined.

**Remark:** Since we usually write the concatenation of paths left to right, we modify the group structure on \( \text{Mod}(F) \) to be \( \psi_1 \bullet \psi_2 = \psi_2 \circ \psi_1 \) so that \( \Psi \) is indeed a homomorphism. This won’t come up much, but being pedantic never hurt anyone.

For a genus \( g \) Lefschetz fibration \( \pi : X \to B \), we define its monodromy representation to be that of the associated fiber bundle \( \pi|_{\pi^{-1}(B^*)} \).

**Remark:** This result is well beyond the scope of this thesis, but in 1996, Matsumoto proved that for genus at least two, the monodromy representation \( \Psi : \pi_1(\Sigma^g) \to \text{Mod}(F) \) completely determines the Lefschetz fibration up to isomorphism, where an isomorphism of Lefschetz fibrations is defined below. The result can even be extended to genus one if the bases spaces have nonempty boundary, or if the sets of critical values are nonempty. For a complete account of this interesting result, the reader can consult [11] [Theorem 2.4].

**Definition 4.5.** An isomorphism of Lefschetz fibrations \( \pi : X \to B \) and \( \pi' : X' \to B' \) is a pair of diffeomorphisms \( f : X \to X' \) and \( g : B \to B' \) which commute with \( \pi \) and \( \pi' \).

Now we’d like to actually describe the monodromy around a singular fiber in some way, i.e., the representation \( \Psi : \pi_1(D^2 \setminus \{p\}) \to \text{Mod}(F) \). However, we note that the monodromy is completely determined by the image of a generator of \( \pi_1(S^1) = \mathbb{Z} \), and so we will often refer to this single diffeomorphism as the monodromy. As it turns out, this diffeomorphism has a very nice geometric interpretation which comes from the corresponding vanishing cycle. To do this, we define the notion of a Dehn twist, complete with some illustrations.

**Definition 4.6.** Suppose that \( F \) is an oriented surface. A right handed Dehn twist \( D_C : F \to F \) on a circle \( C \) in an oriented surface \( F \) is a diffeomorphism obtained by cutting \( F \) along \( C \), twisting \( 360^\circ \) to the right, and regluing. One can view this by means of the diagram below.

![Figure 3: The Dehn twist with \( F \) oriented as a section of a cylinder.](image)

More formally, choose a normal neighbourhood \( N \) of \( C \) in \( F \), and an identification \( \varphi \) of \( N \) with \( S^1 \times I \). Let \( d : S^1 \times I \to S^1 \times I \) be the right handed twist map \( d(\theta, t) = (\theta + 2\pi t, t) \) (we can similarly define a left handed twist). Then the Dehn twist \( D_C : F \to F \) is defined
as $D = \phi^{-1} \circ d \circ \phi$ in $N$, and the identity elsewhere. The twist map $d$ can be visualized in Figure 4 below:

![Figure 4: The twist map $d$ on the annulus.](image)

It turns out that for genus $g$ surfaces, the mapping class group $\text{Mod}(\Sigma_g)$ can be generated solely by Dehn twists. A classical theorem of Dehn in 1920 (proved independently by Lickorish in 1967 [6]) is the following:

**Theorem 4.3** (Dehn-Lickorish). For $g \geq 0$, the mapping class group $\text{Mod}(\Sigma_g)$ is generated by Dehn twists about the $3g - 1$ nonseparating simple closed curves in Figure 5.

**Remark:** In fact, in 1979, Humphries was able to show [6] that $\text{Mod}(\Sigma_g)$ is generated by just the Dehn twists about the Humphries generators $a_1, \ldots, a_{g-1}, c_1, \ldots, c_g, m_1, m_2$, and that $2g + 1$ is the minimal number of generators (if all generators are required to be Dehn twists).

![Figure 5:](image)

These results are not the focus of this thesis, but give context for the appearance of Dehn twists in the Lefschetz fibration construction. For proofs of either of the results above or further reading on mapping class groups one can consult Chapter 4 of [6].

To conclude this section, we will give two properties of Dehn twists that are relatively fundamental and will be used in the next section. Neither result is particularly hard to prove, and we refer the reader to [6].

**Proposition 4.2.** Let $F$ be an oriented surface. Let $\alpha$ and $\beta$ be two isotopy classes of simple closed curves. Then $D_\alpha$ is isotopic to $D_\beta$ if and only if $\alpha = \beta$. In other words, Dehn twists determine their generating cycle up to isotopy.
Proposition 4.3. Let \( F \) be an oriented surface. For any \( \phi \in \text{Mod}(F) \) and any isotopy class of a closed simple curve \( \alpha \), we have \( \phi \circ D_\alpha \circ \phi^{-1} = D_{\phi(\alpha)} \).

4.5 The Monodromy of Lefschetz Fibrations

We are now in a position to describe the monodromy representation of a Lefschetz fibration \( \pi : X \to D^2 \) with just one singular fiber. Unfortunately, the required proof techniques rely on the machinery of handlebody descriptions and Kirby calculus, and so we will not include it. The more interested and knowledgable (or just brave) reader can consult [9] for a discussion of this machinery (in particular, Exercise 8.2.4). In the spirit of moving forward, we’ll just record it as a magnificent fact.

Fact: The monodromy of the Lefschetz fibration \( \pi : X \to D^2 \) with one singular point is given by a right handed Dehn twist about the vanishing cycle for that fiber. [9][Ex. 8.2.4]

Remark: The keen reader may note that while left-handed Dehn twists make perfect sense, they don’t occur as the monodromy of Lefschetz fibrations. This is because our definition of a Lefschetz fibration required orientation-preserving charts. If one relaxes this restriction it is possible to have left-handed twists [7]; these kinds of Lefschetz fibrations are known as achiral, and we will not mention them again.

With this description of the local monodromy of Lefschetz fibrations, we can move on to considering more general Lefschetz fibrations over \( D^2 \). Suppose that \( \pi : X \to D^2 \) is a Lefschetz fibration with \( \mu \) singular fibers at \( p_1, \ldots, p_\mu \in D^2 \). Choose small disjoint disks \( V_1, \ldots, V_\mu \subset \Sigma \) with \( p_i \in V_i \). Then for each \( i \), \( \pi|_{\pi^{-1}(V_i)} \) is a Lefschetz fibration over \( V_i \) and so can be described completely by some vanishing cycle in a nearby regular fiber. We will combine this information to obtain a global description in the following way.

Choose some nonsingular fiber based at \( p_0 \) with another small disjoint disk \( V_0 \) containing \( p_0 \), and choose the labeling of the \( p_i \) to be compatible with some counterclockwise ordering centered at \( p_0 \) for convenience. Choose paths \( s_i \) in the bases from \( p_0 \) to \( p_i \) which are disjoint (except at \( p_0 \)). Since each singular fiber \( F_i = \pi^{-1}(p_i) \) is described by a nearby vanishing cycle on a nonsingular fiber, say \( \gamma_i \), we have a way of transporting these to the common reference fiber \( F_0 \), and hence via some identification to the standard surface \( \Sigma_g \).

In fact, we can prove the following:

Proposition 4.4. For a Lefschetz fibration \( \pi : X \to D^2 \) with \( \mu \) singular fibers as above, the global monodromy about \( \partial D^2 \) is given by the product \( D_{\gamma_1} \cdot \cdots \cdot D_{\gamma_\mu} \), where \( D_{\gamma_i} \) is a Dehn twist about a nearby vanishing cycle \( \gamma_i \) for \( p_i \).

Proof. Let \( C \) be the set of critical values of \( \pi \), and let \( \gamma \in \pi_1(D^2 \setminus C) \) be the curve around
$\partial D^2$. Then $\gamma$ is homotopic to $\gamma'$ as in Figure 6 below.

![Figure 6:](image)

However, we can easily obtain this global monodromy by following $\gamma'$. For each singular fiber $\pi^{-1}(p_i)$, we obtain a monodromy of the form $s^{-1}_i \circ D_{\gamma_i} \circ s_i$, where $D_{\gamma_i}$ is a Dehn twist about some nearby vanishing cycle, and by abuse of notation, we write $s_i \in \text{Mod}(\Sigma_g)$ for the isotopy class of the map $\Sigma_g \to \Sigma_g$ obtained by identifying the images $s_i(0)$ and $s_i(1)$. As noted in the previous section on Dehn twists, the conjugate of a Dehn twist is still a Dehn twist, and so the monodromy about $\gamma'$ is the composition $D_{\gamma'_i} \circ \cdots \circ D_{\gamma'_1}$, where $\gamma'_i$ are vanishing cycles on the regular fiber $\pi^{-1}(p_0)$ (the same cycle up to isotopy).

On the other hand, we see that an ordered collection $D_{\gamma_1}, \ldots, D_{\gamma_\mu}$ of Dehn twists on $\Sigma$ also determines a Lefschetz fibration on $D^2$. Indeed, as noted earlier, a Dehn twist determines its generating circle up to isotopy, so we can always canonically produce a Lefschetz fibration $\pi : X \to D^2$ together with arcs $s_1, \ldots, s_\mu$.

However, while there is a correspondence, we note that Lefschetz fibrations over $D^2$ are not in one to one correspondence with ordered sequences of Dehn twists on $\Sigma_g$. The sequence obtained in Proposition 4.4 depends implicitly on the choices of indices made during the construction, as well as the choice of identification of the regular fiber, and the arcs used. Having reduced the classification of Lefschetz fibrations into a group theoretic problem in the area of mapping class groups, we will do our best to investigate these equivalence classes.

The first ambiguity clearly corresponds to cyclic permutations of $(D_{\gamma_1}, \ldots, D_{\gamma_\mu})$. In a similar fashion, if we change our choice of regular fiber, we will simply conjugate every Dehn twist by some fixed element of $\text{Mod}(\Sigma_g)$. However, this will preserve the overall product, and the net result will just conjugate the product by some element of $\text{Mod}(\Sigma_g)$. The last operation possible is called an elementary transformation, and we will describe it in more detail here. It is not difficult (albeit messy to draw) to see that one can get between different choices of arcs $\{s_i\}$ by way of composing some number of the moves illustrated in Figure 7.

However, via the simple homotopy of curves below, we see that these moves correspond
Figure 7: Elementary transformations.

to changing the ordered collection of Dehn twists \((..., D_{\gamma_i}, D_{\gamma_{i+1}}, ...)\) into \((..., D_{\gamma_{i+1}}, D_{\gamma_{i+1}}^{-1} \ast D_{\gamma_i} \ast D_{\gamma_{i+1}}, ...)\), and so the overall product is also preserved.

Figure 8: The homotopy of paths which shows that \(\gamma_1'(\gamma_2' \gamma_3') \cong \gamma_1 \gamma_3 \gamma_3^{-1} \gamma_2 \gamma_3\).

In fact, these moves completely characterize the correspondence. Another result that we will take as given is the following:

**Fact 4.1.** Two Lefschetz fibrations are isomorphic if and only if it is possible to get between the associated ordered collection of monodromies by way of elementary transformations, together with an inner automorphism of \(\text{Mod}\(\Sigma_g\)\) (i.e., conjugation by some fixed element).

This is proved in a series of exercises in [9][p. 298], but relies again on the machinery of Kirby calculus.

While most of these problems are still intractable by hand, they do lend themselves to computer assisted group theoretic resolutions. We will, however, be able to come to some conclusions in the next section.

### 4.6 Lefschetz Fibrations over \(S^2\)

As mentioned, \(S^2\) provides another compact simply connected base for reasonably simple Lefschetz fibrations. Suppose that \(\pi : X \rightarrow S^2\) is such a fibration. To understand it, we split \(S^2\) as \(D_1^2 \cup D_2^2\), such that \(D_1^2\) contains all the singular fibers of the fibration, say \(F_1, ..., F_\mu\) determined by vanishing cycles \(\gamma_1, ..., \gamma_\mu\), and \(\pi\) is just a genus \(g\) fiber bundle over
Then as noted in the previous section, the monodromy about $\partial D_1^2$ is described by the associated composition of Dehn twists:

$$D_{\gamma_1} \cdots D_{\gamma_n} = D_{\gamma_n} \circ \cdots \circ D_{\gamma_1}$$

On the other hand, this fibration must be trivial on the boundary $\partial D_2^2$, so as to extend to the trivial fibration $\Sigma_g \times D_2^2$. Unfortunately, the ability to extend fibrations isn’t perfect in all genera. With a result of Earle and Eells [4], one can prove [9][p. 300] that the extension is actually unique in genus at least two. In genus one, one can extend the fibration, but two extensions may not be isomorphic. They will, however, be related by multiplicity-1 logarithmic transformations on the last fiber, a fact which we will not give any exposition of, but mention for completeness. In any case, the classification of Lefschetz fibrations over $S^2$ has essentially been reduced to a group theoretic problem in mapping class groups.

Via our correspondence result of the last section, we have [7]:

**Proposition 4.5.** For any fixed $g \geq 2$, there is a one-to-one correspondence between genus $g$ Lefschetz fibrations over $S^2$ and relations of the form $D_{\gamma_1} \cdots D_{\gamma_n}$ in $\text{Mod}(\Sigma_g)$, modulo the relations discussed above, i.e., elementary transformations, and conjugation in $\text{Mod}(\Sigma_g)$.

## 5 Examples

### 5.1 Classification Results

The previous discussion has been strikingly absent of actual examples of Lefschetz fibrations; we will give several now. While we have a characterization of Lefschetz fibrations of $S^2$ as the equivalence classes of trivial words in $\text{Mod}(\Sigma_g)$, this is still rather difficult to classify. We will be able to give a complete classification of all genus one Lefschetz fibrations over $S^2$ (in the sense that we can state it). A partial classification exists in genus two, which is more complicated but still feasible to write down. Not much is known [7] in genera higher than two; unfortunately many known presentations of $\text{Mod}(\Sigma_g)$ have relations which contain both right and left handed twists, so the monodromy approach is less helpful.

### 5.2 The Mapping Class Group of $\Sigma_1$

In this section, we’ll give a short derivation of the mapping class group of $\Sigma_1$.

**Lemma 5.1.** The torus has mapping class group $\text{Mod}(\Sigma_1) = \text{SL}(2, \mathbb{Z})$. 

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Proof. Our sketch of the proof of this result will follow [6]; we’ll leave some of the details to the reader. We define a map

$$\sigma : \text{Mod}(\Sigma_1) \to SL(2, \mathbb{Z})$$

by $$\sigma(\phi) = \phi_*$$, where $$\phi_*$$ is the induced map on $$H_1(\Sigma_1)$$. Indeed, this is well defined since any two representatives of an equivalence class will be homotopic, and so will induce the same map on homology. In fact, since any representative $$\phi$$ is invertible, we have:

$$\phi_* \in \text{Aut}(H_1(\Sigma_1; \mathbb{Z})) = GL(2, \mathbb{Z})$$

We actually have that $$\phi_* \in SL(2, \mathbb{Z})$$, but this follows from an argument involving algebraic intersection numbers that we will leave to the reader [6].

To prove that $$\sigma$$ is onto, suppose that $$M \in SL(2, \mathbb{Z})$$. Then $$M$$ is certainly a linear homeomorphism compatible with the $$\mathbb{Z}^2$$ lattice in $$\mathbb{R}^2$$, and so we obtain a map on $$\Sigma_1$$, say $$\phi_M$$, and $$\sigma([\phi_M]) = M$$.

To prove that $$\sigma$$ is injective, we suppose that $$\sigma([\phi]) = I$$. Let $$\alpha$$ and $$\beta$$ be representatives of the homotopy classes defined by the vectors $$[1, 0]$$ and $$[0, 1]$$, i.e., the longitude and meridian loops. Then we have $$\phi(\alpha) \sim \alpha$$ and $$\phi(\beta) \sim \beta$$ (where $$\sim$$ denotes homotopy equivalence), and by standard results (Prop 1.10, 1.11 of [6]), this homotopy extends to an isotopy, which itself extends a map on $$\Sigma_1$$. But this shows that $$\phi$$ is isotopic to the identity map on $$\Sigma_1$$, so $$\sigma$$ is injective.

Remark: We recall that $$SL(2, \mathbb{Z})$$ has a relatively nice group presentation: it can be presented as:

$$SL(2, \mathbb{Z}) = \langle \alpha_1, \alpha_2; \alpha_1^4 = 1, \alpha_2^2 = (\alpha_1 \alpha_2)^3 \rangle$$

where $$\alpha_1$$ and $$\alpha_2$$ are identified via:

$$\alpha_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

One can easily check that these relations hold, but to show that $$\alpha_1$$ and $$\alpha_2$$ actually generate $$SL(2, \mathbb{Z})$$ is somewhat computation heavy. More importantly, these generators and relations correspond to the “correct” loops in $$\Sigma_1$$, in the sense that $$\alpha_1$$ and $$\alpha_2$$ are identified with right handed Dehn twists about the curves drawn below.

This result is fairly well known, so we’ll omit it.
5.3 Genus One

Having computed \( \text{Mod}(\Sigma_1) \) above, we note that the relation \( (\alpha_1\alpha_2)^6n \) holds, and so for each \( n \) we have a Lefschetz fibration \( \pi_n : X_n \to S^2 \). The careful reader will note that there may be more than one, since we were only guaranteed unique extensions to \( S^2 \) when \( g \geq 2 \). However, this turns out not to be a problem, as Moishezon proved [7] that the global monodromy of any nontrivial genus one Lefshetz fibration is in fact equivalent (modulo the allowed moves) to one of these relations, and so any elliptic fibration is isomorphic to some \( \pi_n \).

On the other hand, we have a concrete description of elliptic surfaces via algebraic geometry. One can consider two generic cubics \( p_0 \) and \( p_1 \) on \( \mathbb{CP}^2 \), which intersect in 9 points, say \( P_1, \ldots, P_9 \). Then we can define a map \( f : \mathbb{CP}^2 \setminus \{P_1, \ldots, P_9\} \to \mathbb{CP}^1 \) in the following way: for \( Q \in \mathbb{CP}^2 \setminus \{P_1, \ldots, P_9\} \), take the unique cubic \( t_0p_0 + t_1p_1 \) passing through \( Q \) (by standard lemmas) and set \( f(Q) = [t_0, t_1] \). We can’t define \( f \) at \( P_1, \ldots, P_9 \), but by blowing up \( \mathbb{CP}^2 \) each point, we can extend \( f \) to a fibration \( \pi : \mathbb{CP}^2 \# 9\mathbb{CP}^2 \to \mathbb{CP}^1 \). We usually denote \( E(1) = \mathbb{CP}^2 \# 9\mathbb{CP}^2 \) when we endow it with this fibration structure. This works with polynomials of different degree, so we summarize the situation with the following lemma from [9][p. 69]:

**Lemma 5.2.** The manifold \( \mathbb{CP}^2 \# d\mathbb{CP}^2 \) admits a singular fibration to \( \mathbb{CP}^1 = S^2 \), where the generic fiber is a complex curve of genus \( \frac{1}{2}(d-1)(d-2) \).

**Remark:** We note that if any fibration \( \pi : X \to S^2 \) had all fibers diffeomorphic to \( \Sigma_1 \), we would have \( \chi(X) = \chi(S^2)\chi(\Sigma_1) = 0 \). On the other hand, we can calculate the Euler characteristic of \( E(1) \) using the connected sum formula from Section 3.2 (\( \mathbb{CP}^2 \) is compact and connected):

\[
\chi(\mathbb{CP}^2 \# \mathbb{CP}^2) = \chi(\mathbb{CP}^2) + \chi(\mathbb{CP}^2) - \chi(S^4) = 3 + 3 - 2 = 4
\]

\[
\chi([\mathbb{CP}^2 \# \mathbb{CP}^2] \# \mathbb{CP}^2) = \chi(\mathbb{CP}^2 \# \mathbb{CP}^2) + \chi(\mathbb{CP}^2) - \chi(S^4) = 4 + 3 - 2 = 5
\]

After 7 more iterations, we find \( \chi(E(1)) = 12 \), and so \( E(1) \) must have fibers which are not diffeomorphic to \( \Sigma_1 \).
We can also produce more elliptic fibrations with an operation called the fiber sum.

**Definition 5.1.** Let \( \pi : X \to C \) and \( \pi' : X' \to C' \) be two genus \( g \) fibrations, and let \( F \subset X, F' \subset X \) be two regular fibers. Identify neighbourhoods of \( F \) and \( F' \) with \( F \times D^2 \) and \( F' \times D^2 \), and choose any diffeomorphism \( h : F \to F' \). The fiber sum \( X \# F X \) is the manifold \( (X \setminus F \times D^2) \cup \psi (X' \setminus F' \times D^2) \), where \( \psi : \partial (F \times D^2) \to \partial (F' \times D^2) \) is given by \( \psi = h \times \) (complex conjugation) : \( F \times S^1 \to F' \times S^1 \).

While this does depend on several choices made during the construction, it does produce a unique object in the following case [9][p. 72].

**Definition 5.2.** The elliptic surface \( E(n) \) is defined inductively as \( E(n) := E(n-1) \# F E(1) \).

In fact, while the following fact is not obvious, it is true [9][p. 72].

**Fact 5.1.** \( E(n) \) can be given a complex structure. Since complex manifolds always come with a canonical choice of orientation, this implies that \( E(n) \) is orientable for each \( n \).

Hence, by our discussion above, we have two different descriptions of \( E(n) \) (we know we have a nice correspondence between \( E(n) \) and \( X_n \) above by computing Euler characteristics, for example [9][p. 74]). The first doesn’t rely on any algebraic geometry, which is rather nice. We’ll finish our discussion of these spaces by computing the rest of the invariants of \( E(1) \).

### 5.4 Topological Invariants of \( E(1) \)

The invariants of \( E(n) \) are computable, but not by particularly elementary methods. We’ll compute the invariants of \( E(1) \), since this doesn’t require any further understanding of the fiber sum. Since \( \mathbb{CP}^2 \) is closed and connected, it follows that \( E(1) \) is as well. By the fact above, it is also orientable, and so its homology and cohomology has the form described in lemma 3.1. We recall the two following standard facts:

**Proposition 5.1.** \( \mathbb{CP}^n \) is simply connected for each \( n \).

**Proposition 5.2.** If \( M_1, M_2 \) are two connected \( m \)-manifolds with \( m \geq 3 \), then \( \pi_1(M_1 \# M_2) = \pi_1(M_1) \star \pi_1(M_2) \), where \( \star \) denotes the free product.

This implies that \( \pi_1(E(1)) = \pi_1(\mathbb{CP}^2) \star \cdots \star \pi_1(\mathbb{CP}^2) = 1 \), and so \( E(1) \) is simply connected.

Then we have:
by lemma 3.1. Since $E(1)$ is simply connected, we also get:

$$H_1(E(1)) \cong 0$$
$$H_3(E(1)) \cong 0$$

and lastly since $H_2(E(1))$ is free and $\chi(E(1)) = 12$:

$$H_2(E(1)) \cong \mathbb{Z}^{12-2} = \mathbb{Z}^{10}$$

By Poincaré duality, the cohomology modules are:

$$H^0(E(1)) \cong \mathbb{Z}$$
$$H^1(E(1)) \cong 0$$
$$H^2(E(1)) \cong \mathbb{Z}^{10}$$
$$H^3(E(1)) \cong 0$$
$$H^4(E(1)) \cong \mathbb{Z}$$

To get the signature of $E(1)$, we use the results in Section 3.5. Since $\sigma(\mathbb{CP}^2) = 1$, $\sigma(\overline{\mathbb{CP}^2}) = -1$, and signature is additive under connected sums, we obtain:

$$\sigma(E(1)) = \sigma(\mathbb{CP}^2) + 9\sigma(\overline{\mathbb{CP}^2}) = -8$$

5.5 A Higher Genus Example

Unfortunately, higher genus examples are much more complicated, and less is currently known. However, there have been some recent partial classification results [7]. Let $D_1, \ldots, D_{2g+1}$ be right handed Dehn twists about the non-separating curves $\alpha_1, \ldots, \alpha_{2g+1}$ in the figure below.

Then one can prove (with considerable effort) that the following relations hold in $\text{Mod}(\Sigma_g)$ (we’ll omit the $\ast$ notation and just write composition as just words in $\text{Mod}(\Sigma_g)$):
These correspond to Lefschetz fibrations over $S^2$, which, following [7], we’ll denote by $X(1), X(2), X(3)$. Each can be shown to be complex. In genus two, Chakiris [3] (and later Smith [13]) were able to prove that any holomorphic Lefschetz fibration with only non-separating cycles is a fiber sum of one of these three, which is certainly interesting progress towards a complete classification.
References