PMath 990, Fall Term 2017

Free analogues for fundamental probabilistic structures

Day/Time/Room: Tu Th 4–5:20 pm, in MC 1085

Instructor: Andu Nica (anica@uwaterloo.ca)

Course outline

This course examines how "free, non-commutative" analogues are found for some fundamental structures of probability theory, and for some classical theorems concerning these structures. This development, the so-called *free probability*, was started in the 1980's and has since then accumulated a rather extensive body of work. The goal of the course is to get the students to understand and appreciate a few basic items in the dictionary of "classical probabilistic vs free probabilistic" terms which was created by the above mentioned body of work. For every such item, our approach will be to first review (with details as needed by the background of the students enrolled in the class) the classical probabilistic concept which is under scrutiny, and then to discuss its free probabilistic counterpart.

The required background for the course is a good knowledge of basic analysis in abstract framework, including measure and integration, and including some rudimentary facts about bounded linear operators on a Hilbert space. (Having a more advanced knowledge of either classical probability theory or of operator algebras may lead to a better appreciation of the topic, but is not a prerequisite.)

More details on the course outline

In reference to the dictionary of classical vs free probabilistic terms that was mentioned above, here are some of the "dictionary items" which we want to discuss.

• We will start from the classical concepts of what is a real random variable (a.k.a. measurable function), and of what it means for a family of real random variables to be independent. The cornerstone of free probability is the parallel concept of *freely independent family of non-commutative self-adjoint random variables* (where it should be noted that a non-commutative random variable isn't generally a function, one should think of it as a matrix, or more generally as an operator on a Hilbert space).

• A real random variable has a *distribution*, which is a probability measure on $\mathbb{R}$, and the operation of adding independent random variables corresponds to the operation of classical convolution for probability measures on $\mathbb{R}$. Free analogue: a non-commutative self-adjoint random variable has a distribution which is (still) a probability measure on $\mathbb{R}$; the addition of freely independent random variables corresponds to an operation with probability measures called *free convolution*, usually denoted as $\boxplus$.

• In order to study classical convolution of probability measures on $\mathbb{R}$, one resorts to the Fourier transform (a.k.a. characteristic function) of such probability measures. In order to study the free convolution $\boxplus$, one uses a device called *the R-transform*.

• The most important distribution in classical probability is the Gaussian (or normal) distribution, as one sees by looking at the fundamental result called "central limit theorem".

In order to determine what is the free analogue of the Gaussian, we will prove the *free central limit theorem*, and we will find that the limit distribution arising there is the so-called *semicircle law of Wigner*.

• The next most important distribution in classical probability is the Poisson distribution, which is found as limit in the Poisson limit theorem. In order to see what is the free analogue of this, we will prove the *free Poisson limit theorem*, and we will find that the limit distribution arising there is the so-called *Marchenko-Pastur distribution*. (Same as is the case for Wigner's law, the Marchenko-Pastur distribution was first identified in work about random matrices; the re-occurrence of these same distributions in free probability is more than a coincidence, but the connection to random matrices will not be pursued in this course.)

• A fundamental concept in classical probability is the one of a process with time parameter in $[0, \infty)$. We will look at such processes that are stationary and have independent increments; the two most important examples here are the Brownian motion (has Gaussian increments) and the Poisson process (has increments with Poisson distribution). Free analogue: we will look at stationary processes with *freely independent increments*; among them we will find the *free Brownian motion* (has increments with semicircular distributions) and the *free Poisson process* (has increments with Marchenko-Pastur distributions).

References

I will try to make the lectures as self-contained as possible, but for additional reading and for course-projects there exist several textbooks of introduction to free probability that are available:

• G.W. Anderson, A. Guionnet, O. Zeitouni. *An introduction to random matrices*, Cambridge University Press, 2009. (The final part of this book gives a general introduction to free probability.)

• J.A. Mingo, R. Speicher. *Free probability and random matrices*, Springer, 2016.

• A. Nica, R. Speicher. *Lectures on the combinatorics of free probability*, Cambridge University Press, 2006. (The first part of this book gives a general introduction to free probability.)

For the considerations on processes with free increments, we will have to also look directly at some research papers on this topic, such as:

• P. Biane. Processes with free increments, *Mathematische Zeitschrift* 227 (1998), 143-174.

• P. Biane, R. Speicher. Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, *Probability theory and related fields* 112 (1998), 373-409.

The classical probability aspects discussed in the course are covered by most of the "standard" monographs used as reference for this area, for instance:

• R.M. Dudley. *Real analysis and probability*, Cambridge University Press, 2002.

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