A note on large gaps between consecutive prime numbers

by C.L. Stewart*

In 1938 Rankin [7], improving on an earlier result of Erdös [3], showed that for each $\varepsilon > 0$ there exist infinitely many integers n such that

$$p_{n+1}-p_n > (1/3 - \varepsilon)\log p_n \log_2 p_n \log_4 p_n/(\log_3 p_n)^2, \qquad (1)$$

where p_n denotes the n-th largest prime number and $\log_i = \log(\log_{i-1})$ denotes the i-th iteration of the logarithm function. In 1963 Schönhage [9] showed that it is possible to replace $1/3$ in (1) by $e^\gamma/2$ where γ is Euler's constant and Rankin [8] in the same year improved $e^\gamma/2$ to e^γ $(= 1.78\ldots)$. In this note I would like to give a simple proof of (1) with $1/3$ replaced by $1/2$.

Let t be a positive integer and let ε be a real number with $0 < \varepsilon < 1/2$. Denote the interval $[t^{(1-\varepsilon)\log_3 t/\log_2 t}, t/\log t]$ by T and put

$$k = \prod_{p\in T} p,$$

where the product is taken over primes p. Note that by the prime number theorem

$$k = e^{(1+o(1))t/\log t}. \qquad (2)$$

The number of integers from $1,\ldots,t$ which are coprime with k is at most N_1+N_2 where N_1 is the number of positive integers less than or equal to t all of whose prime factors are less than $t^{(1-\varepsilon)\log_3 t/\log_2 t}$ and where N_2 is the number of positive integers

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less than or equal to t and having a prime factor greater than $t/\log t$. It follows from work of de Bruijn, in particular (1.3) and (1.4) of [1] and (1.8) of [2], that $N_1 = t/u^{u(1+o(1))}$ with $u = \log_2 t/(1-\varepsilon)\log_3 t$, hence $N_1 = o(t/\log t)$. Further,

$$N_2 = \sum_{t/\log t < p \le t} [\tfrac{t}{p}] \le t \sum_{t/\log t < p \le t} \frac{1}{p},$$

hence, (by 22.7.3 and 22.7.4 of [4]),

$$\begin{aligned} N_2 &\le t(\log_2 t - \log(\log t - \log_2 t) + O(1/\log t)) \\ &\le t(-\log(1-(\log_2 t/\log t)) + O(1/\log t)) \\ &\le t \log_2 t/\log t + O(t/\log t). \end{aligned}$$

Therefore

$$N_1 + N_2 \le (1+o(1))t \log_2 t/\log t. \tag{3}$$

Let S denote the number of primes which are of the form $kz+\ell$ with $1 \le \ell \le t$ and $1 \le z \le k^{[\log_2 t]}$. By the Brun-Titchmarsh theorem (see Theorem 2 of [6]),

$$S \le (N_1+N_2)2k^{1+[\log_2 t]} / \phi(k)\log k^{[\log_2 t]}. \tag{4}$$

Since $\phi(k) = k \prod_{p\varepsilon T} (1-\frac{1}{p})$ we have, by Mertens' theorem (Theorem 429 of [4]),

$$\phi(k) = k(1-\varepsilon + o(1))\log_3 t/\log_2 t. \tag{5}$$

We find, from (2),(3),(4) and (5), that

$$S \le (2/(1-\varepsilon)+o(1))(\log_2 t/\log_3 t)k^{[\log_2 t]}.$$

Thus for some integer z_0 with $1 \le z_0 \le k^{[\log_2 t]}$ the interval $[kz_0+1, kz_0+t]$ will contain at most $(2/(1-\varepsilon)+o(1))(\log_2 t/\log_3 t)$ primes and so in this interval there will be a gap between consecutive primes, p_n and p_{n+1} say, of size at least $((1-\varepsilon)/2+o(1))t \log_3 t/\log_2 t$. But $p_n \le k^{1+[\log_2 t]}+t$ hence, recall (2),

$$\log p_n \le (1+o(1))t \log_2 t/\log t, \tag{6}$$

and so,

$$p_{n+1}-p_n \ge (\tfrac{1}{2} - \tfrac{\varepsilon}{2} + o(1))\log p_n \log t \log_3 t/(\log_2 t)^2.$$

Our result now follows since, by (6), $\log_2 p_n \le (1+o(1))\log t$.

The key estimate in the above argument is the estimate for N_1 which allows us to sieve the integers from $1,\dots,t$ by primes from T in an efficient manner. A similar estimate also appears in the proofs of Rankin [7], [8] and of Schönhage [9]. Rankin and Schönhage construct, by a sieving process and the Chinese Remainder Theorem, long intervals free of primes whereas we use the Brun-Titchmarsh theorem and an averaging argument to this end. Maier [5] has used a related averaging argument to show that for each positive integer j there is a positive number c_j such that for infinitely many integers n,

$$p_{n+i+1}-p_{n+i} > c_j \log p_n \log_2 p_n \log_4 p_n/(\log_3 p_n)^2,$$

for $i = 0,\dots,j-1$.

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