SETS GENERATED BY FINITE SETS OF ALGEBRAIC NUMBERS

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For Professor Robert Tijdeman on the occasion of his seventy-fifth birthday.

1. Introduction

Let \( S = \{p_1, \ldots, p_r\} \) be a finite set of prime numbers with \( r \geq 2 \) and let \( (n_i)^\infty_{i=1} \) be the increasing sequence of positive integers composed of the primes from \( S \). In 1973 [10] and 1974 [11] Tijdeman proved that there exist positive numbers \( c_1, c_2 \) and \( c_3 \), effectively computable in terms of \( S \), such that for \( n_i \geq c_3 \)

\[
\frac{n_i}{\log n_i} < n_{i+1} - n_i < \frac{n_i}{\log n_i} c_2.
\]

Tijdeman [10] also resolved a question of Wintner by proving that there exist infinite sets of primes \( S \) for which the associated sequence \( (n_i)^\infty_{i=1} \) satisfies

\[
\lim_{i \to \infty} n_{i+1} - n_i = \infty,
\]

see also [5].

In this note we shall study the distribution of the numbers formed when we take \( S \) to be a finite set of multiplicatively independent algebraic numbers of absolute value larger than 1 instead of a finite set of primes. Our first result corresponds to the lower bound in (1) and shows that such numbers are not close to each other.

**Theorem 1.** Let \( \alpha_1, \ldots, \alpha_r \) be multiplicatively independent algebraic numbers with \( |\alpha_i| > 1 \) for \( i = 1, \ldots, r \). Put

\[
T = \{\alpha_1^{h_1} \cdots \alpha_r^{h_r} \mid h_i \geq 0 \text{ for } i = 1, \ldots, r\}.
\]

There exists a positive number \( c \), which is effectively computable in terms of \( \alpha_1, \ldots, \alpha_r \), such that if \( t \) and \( t' \) are in \( T \) with \( |t| \geq 3 \) then

\[
|t - t'| > |t|/(\log |t|)^c.
\]

Theorem 1 follows directly from lower bounds for linear forms in the logarithms of algebraic numbers, see [1, 2, 3, 6, 7].

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We next obtain generalizations of the upper bound in (1). We consider two cases. For the first case we restrict our attention to sets of real algebraic numbers.

**Theorem 2.** Let \( \alpha_1 \) and \( \alpha_2 \) be multiplicatively independent real algebraic numbers. Suppose \( \alpha_i > 1 \) for \( i = 1, 2 \) and put

\[
T = \{ \alpha_1^{h_1} \alpha_2^{h_2} \mid h_i \geq 0 \text{ for } i = 1, 2 \}. 
\]

There exists a positive number \( c_1 \), which is effectively computable in terms of \( \alpha_1 \) and \( \alpha_2 \), such that for any real number \( x \) with \( x \geq 3 \) there exists an element \( t \) of \( T \) with

\[
| x - t | < x/(\log x)^{c_1}.
\]

For the proof of Theorem 2 we modify the argument given by Tijdeman in [11].

Finally we consider the case when the elements of \( T \) are not all real.

**Theorem 3.** Let \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) be multiplicatively independent algebraic numbers with \( |\alpha_i| > 1 \) for \( i = 1, 2, 3 \). Suppose that \( \alpha_1 \) and \( \alpha_2 \) are positive real numbers and that \( \alpha_3/|\alpha_3| \) is not a root of unity. Put

\[
T = \{ \alpha_1^{h_1} \alpha_2^{h_2} \alpha_3^{h_3} \mid h_i \geq 0 \text{ for } i = 1, 2, 3 \}. 
\]

There exists a positive number \( c_2 \), which is effectively computable in terms of \( \alpha_1, \alpha_2 \) and \( \alpha_3 \), such that for any complex number \( z \) with \( |z| \geq 3 \) there exists an element \( t \) of \( T \) with

\[
| z - t | < |z|/(\log |z|)^{c_2}.
\]

Observe that if \( \alpha_1 \) and \( \alpha_2 \) are real numbers and \( \alpha_3/|\alpha_3| \) is a root of unity then there is a positive number \( c_4 \) and complex numbers \( z \) of arbitrarily large modulus for which

\[
| z - t | > c_4|z|
\]

for all elements \( t \) in \( T \).

With Min Sha and Igor Shparlinski [8] we have applied both (1) and Theorem 3 in order to study the distribution of multiplicatively dependent vectors of algebraic numbers.

2. **Linear forms in the logarithms of algebraic numbers**

For any algebraic number \( \alpha \) the height of \( \alpha \) is the maximum of the absolute values of the relatively prime integer coefficients of the minimal polynomial of \( \alpha \). Let \( \alpha_1, \ldots, \alpha_n \) be algebraic numbers of heights at most
$A_1, \ldots, A_n$ respectively. Let $b_1, \ldots, b_n$ be non-zero integers of absolute value at most $B$ with $B \geq 2$. Put

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n$$

and

$$d = [Q(\alpha_1, \ldots, \alpha_n) : Q].$$

Baker [1, 2] and Feldman [3] proved the following result.

**Lemma 4.** There is a positive number $c$, which depends on $A_1, \ldots, A_n, n$ and $d$, such that if $\Lambda \neq 0$ then

$$|\Lambda| > B^{-c}.$$

For a sharp explicit dependence of $c$ in Lemma 4 on the parameters $A_1, \ldots, A_n, n$ and $d$, see Matveev [6, 7].

3. **Proof of Theorem 1**

Let $c_1, c_2, \ldots$ be positive numbers which are effectively computable in terms of $\alpha_1, \ldots, \alpha_r$. Let $t$ be in $T$ with $|t| \geq 3$. Then

$$t = \alpha_1^{h_1} \cdots \alpha_r^{h_r}$$

with $h_i \geq 0$ for $i = 1, \ldots, r$. Suppose $t'$ is in $T$ with $t' \neq t$. We have

$$t' = \alpha_1^{j_1} \cdots \alpha_r^{j_r}$$

with $j_i \geq 0$ for $i = 1, \ldots, r$.

Then

$$|t - t'| = |t| |\alpha_1^{j_1 - h_1} \cdots \alpha_r^{j_r - h_r} - 1|.$$

Since $t \neq t'$ we may apply Lemma 4, as in Theorem A of [9], to give

(2) $$|t - t'| > |t| B^{-c_1},$$

where

$$B = \max(4, |j_1 - h_1|, \ldots, |j_r - h_r|).$$

We may suppose that $|t'| \leq 2|t|$ since otherwise the result holds and thus

(3) $$B < c_2 \log |t|.$$

Our result now follows from (2) and (3) since $|t| \geq 3$. 

4. A preliminary result for the proof of Theorem 2

Let \( \alpha_1 \) and \( \alpha_2 \) be multiplicatively independent real algebraic numbers with \( \alpha_i > 1 \) for \( i = 1, 2 \). Let \( \ell_0/k_0, \ell_1/k_1, \ldots \) be the sequence of convergents to \( \log \alpha_1/\log \alpha_2 \). Our next result gives a bound on the growth of the \( k_i \)'s. The proof depends upon Lemma 4 and is due to Tijdeman [11] when \( \alpha_1 \) and \( \alpha_2 \) are distinct primes.

**Lemma 5.** There exists a positive number \( c \), which is effectively computable in terms of \( \alpha_1 \) and \( \alpha_2 \), such that

\[
k_{j+1} < k_j^c
\]

for \( j = 2, 3, \ldots \).

**Proof.** Replacing \( \log p/\log q \) by \( \log \alpha_1/\log \alpha_2 \) in the proof of the Lemma in [11] and noting that \( k_j \geq 2 \) for \( j \geq 2 \) we obtain the result. \( \square \)

5. Proof of Theorem 2

Let \( c_1, c_2, \ldots \) denote positive numbers which are effectively computable in terms of \( \alpha_1 \) and \( \alpha_2 \). Let \( x \) be a real number with \( x \geq 3 \) and let \( t \) be the largest element of \( T \) with \( t \leq x \). Then

\[
x \leq \max(\alpha_1, \alpha_2)
\]

and so

\[
\frac{1}{2} \log x < \log t,
\]

for \( x > c_1 \).

We have

\[
t = \alpha_1^{h_1} \alpha_2^{h_2}
\]

with \( h_1 \) and \( h_2 \) non-negative integers. We may assume, without loss of generality, that

\[
\alpha_1^{h_1} \geq t^{1/2}
\]

and so

\[
h_1 \geq \frac{1}{2 \log \alpha_1} \log t.
\]

Let \( \ell_0/k_0, \ell_1/k_1, \ldots \) be the sequence of convergents from the continued fraction expansion of \( \log \alpha_1/\log \alpha_2 \). Recall that the convergents with even index are smaller that \( \log \alpha_1/\log \alpha_2 \) and those with odd index are larger. Choose \( j \) to be the largest odd integer for which

\[
k_j \leq h_1;
\]
certainly \( k_1 \leq h_1 \) for \( x > c_2 \). Then \( k_{j+2} \) exceeds \( h_1 \) and by (4) and (5)

\[
k_{j+2} > \frac{1}{4 \log \alpha_1} \log x
\]

for \( x > c_3 \). By Lemma 5, \( k_{j+2} < k_{j+1}^{c_4} \) and so

\[
k_{j+1} > \left( \frac{1}{4 \log \alpha_1} \log x \right)^{1/c_4}
\]

hence

\[(7) \quad k_{j+1} > (\log x)^{c_5}\]

for \( x > c_6 \).

Put

\[
t' = \alpha_1^{h_1-k_j} \alpha_2^{h_2+\ell_j}
\]

and note that by (6) \( t' \) is in \( T \). Further since \( t < t' \) we have \( x < t' \). By Theorem 167 and Theorem 171 of [4],

\[
0 < \frac{\ell_j}{k_j} - \frac{\log \alpha_1}{\log \alpha_2} < \frac{1}{k_j k_{j+1}}
\]

and so

\[(8) \quad \log(t'/t) < \frac{\log \alpha_2}{k_{j+1}}.
\]

It follows from (7) and (8) that \( \log(t'/t) < \frac{1}{4} \) hence

\[(9) \quad \log(t'/t) = \log(1 + (t' - t)/t) > \frac{t' - t}{2t}
\]

and thus, by (8) and (9),

\[(10) \quad t' - t < \frac{(2 \log \alpha_2) t}{k_{j+1}},
\]

for \( x > c_6 \). Recall (7) and that \( t \leq x < t' \). We see from (10) that

\[
x - t < c_7 \frac{x}{(\log x)^{c_8}} < \frac{x}{(\log x)^{c_9}}
\]

for \( x > c_{10} \). Note that this suffices to prove Theorem 2 since

\[
x - t \leq x - 1 \leq \frac{x}{(1 + \frac{1}{x-1})} \leq \frac{x}{(\log x)^{c_{11}}}
\]

for \( 3 \leq x \leq c_{10} \).
6. Preliminaries for the proof of Theorem 3

For any non-zero complex number \( z \) let \( \text{Im}(z) \) denote the imaginary part of \( z \), let \( \text{Arg}(z) \) denote the argument of \( z \) chosen so that \( 0 \leq \text{Arg}(z) < 2\pi \) and let \( \log z \) denote the principal branch of the logarithm so that \( 0 \leq \text{Im}(\log z) < 2\pi \).

Let \( \nu \) be a real number with \( 0 \leq \nu < 2\pi \) and let \( \alpha \) be an algebraic number with \( |\alpha| > 1 \) for which \( \alpha/|\alpha| \) is not a root of unity. For each positive integer \( k \) let \( b_k \) be the smallest positive integer for which

\[
|\text{Arg}(\alpha^{b_k}) - \nu| \leq \frac{2\pi}{k}.
\]

**Lemma 6.** There exists a positive number \( c \), which is effectively computable in terms of \( \alpha \), such that

\[
b_k < k^c
\]

for \( k \geq 2 \).

**Proof.** Suppose \( k \geq 2 \). By Dirichlet’s box principle there exists an integer \( r_k \) with \( 1 \leq r_k \leq k \) and an integer \( m_k \) so that

\[
\left| r_k \log \frac{\alpha}{|\alpha|} - m_k 2\pi i \right| \leq \frac{2\pi}{k}.
\]

Notice that we have \( 0 \leq m_k \leq r_k \).

Let \( c_1, c_2, \ldots \) denote positive numbers which are effectively computable in terms of \( \alpha \). By Lemma 4

\[
\left| r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1) \right| \geq \frac{1}{(2r_k)^{c_1}} \geq \frac{1}{k^{c_2}}.
\]

If

\[
0 < \frac{1}{i} \left( r_k \log \frac{\alpha}{|\alpha|} - m_k 2\pi i \right) \leq \frac{2\pi}{k}
\]

then there exists an integer \( q_k \) with \( 1 \leq q_k \leq 2\pi k^{c_2} \) for which

\[
\left| q_k \left( r_k \log \frac{\alpha}{|\alpha|} - m_k \log(-1) \right) - \nu i \right| \leq \frac{2\pi}{k}.
\]

On the other hand if \( r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1) \) is of the form \( yi \) with \(-\frac{2\pi}{k} \leq y < 0\) then there exists an integer \( q_k \) with \( 1 \leq q_k \leq 2\pi k^{c_2} \) for which

\[
\left| 2\pi i + q_k \left( r_k \log \frac{\alpha}{|\alpha|} - m_k \log(-1) \right) - \nu i \right| \leq \frac{2\pi}{k}.
\]

Therefore, since

\[
\log\left( \frac{\alpha}{|\alpha|} \right)^{qr_k} = i\text{Arg}(\alpha^{qr_k})
\]

and since \( k \geq 2 \),

\[
b_k \leq qr_k \leq 2\pi k^{1+c_2} < k^{c_3}
\]

as required. \( \Box \)
7. Proof of Theorem 3

Let \( z \) be a complex number with \( |z| \geq 3 \) and put \( \nu = \text{Arg}(z) \). Let \( b_1, b_2, \ldots \) be defined as in \( \S 6 \) with \( \alpha \) replaced by \( \alpha_3 \). Let \( c_1, c_2, \ldots \) denote positive numbers which are effectively computable in terms of \( \alpha_1, \alpha_2 \) and \( \alpha_3 \). It suffices, as in the proof of Theorem 2, to establish our result for \( |z| > c_1 \). In particular we may suppose that \( |z| \) exceeds \( \max(9, |\alpha_3|^{2b_2}) \). We now choose \( k \) so that

\[
|\alpha_3|^{2b_k} \leq |z| < |\alpha_3|^{2b_{k+1}};
\]

since \( |z| \) exceeds \( |\alpha_3|^{2b_2} \) and since the sequence \( (b_i)_{i=1}^{\infty} \) is non-decreasing \( k \) is well defined. By the definition of \( b_k \) we have

\[
|\text{Arg}(\alpha_3^{b_k}) - \text{Arg}(z)| \leq \frac{2\pi}{k}.
\]

Further

\[
3 \leq |z|^{1/2} \leq |z|/|\alpha_3|^{b_k} < |z|.
\]

We now choose non-negative integers \( j_1 \) and \( j_2 \) such that

\[
|\alpha_1^{j_1} \alpha_2^{j_2} - \frac{|z|}{|\alpha_3|^{b_k}}| < \frac{|z|/|\alpha_3|^{b_k}}{(\log |z|)^{c_2}}
\]

which is possible by (12) and Theorem 2. Put

\[ t = \alpha_1^{j_1} \alpha_2^{j_2} \alpha_3^{b_k}. \]

Notice that by (13)

\[
||z| - |t|| < \frac{|z|}{(\log |z|)^{c_2}}.
\]

In addition, since \( \text{Arg}(t) = \text{Arg}(\alpha_3^{b_k}) \),

\[
|\text{Arg}(z) - \text{Arg}(t)| \leq \frac{2\pi}{k}.
\]

On the other hand, by (11),

\[
\log |z| < 2b_{k+1} \log |\alpha_3|
\]

and so by Lemma 6

\[
|z| < k^{c_3}
\]

since \( k \geq 2 \). Thus by (15) and (16)

\[
|\text{Arg}(z) - \text{Arg}(t)| < \frac{2\pi}{(\log |z|)^{c_4}}.
\]

Our result now follows from (14) and (17).
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References
