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On intervals with few prime numbers

In memory of Professor Paul Erdős

By *H. Maier* at Ulm and *C. L. Stewart* at Waterloo

Abstract. We prove that there are long intervals containing fewer prime numbers than the average for intervals of such length.

1. Introduction

The first proof that there exist gaps between consecutive prime numbers which are much larger than the average was given by Westzynthius [29] in 1931. Let $c_0, c_1, \ldots$ denote positive constants. He proved that for arbitrarily large integers x,

$$\pi\bigl(x + \Phi(x)\bigr) - \pi(x) = 0, \tag{1.1}$$

with

$$\Phi(x) = c_0 \frac{\log x \log_3 x}{\log_4 x}, \tag{1.2}$$

where $\log_i = \log(\log_{i-1})$ denotes the i-th iteration of the logarithm function and $\pi(x)$ denotes the number of primes less than or equal to x. In 1934 Ricci [25] removed the factor of $\log_4 x$ from the denominator of (1.2). One year later Erdős [10] established that (1.1) holds for infinitely many integers x with

$$\Phi(x) = c_1 \frac{\log x \log_2 x}{(\log_3 x)^2}.$$

In 1938 Rankin [23] showed that this result also could be improved by a factor of $\log_4 x$ and so (1.1) holds with

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$$\Phi(x) = c_2 \frac{\log x \log_2 x \log_4 x}{(\log_3 x)^2}, \tag{1.3}$$

for arbitrarily large integers x. Rankin proved that (1.3) holds with c_2 any positive real number less than $1/3$. Subsequent improvements by Schönhage [26], Rankin [24], Maier and Pomerance [18] and Pintz [22] have concerned the value of c_2. The best result to date is due to Pintz [22] who proved that c_2 may be taken to be any positive real number less than $2e^{\gamma}(= 3.5621\ldots)$.

One might expect, from a consideration of the prime number theorem, that if $\Phi(x)$ grows sufficiently quickly as a function of x, then

$$\pi\bigl(x + \Phi(x)\bigr) - \pi(x) \sim \frac{\Phi(x)}{\log x}, \tag{1.4}$$

as $x \to \infty$. In 1943, Selberg [27], under the assumption of the Riemann hypothesis, proved that (1.4) holds for almost all x, in the sense of Lebesgue measure, provided that $\Phi(x)$ is positive and increasing and that $\Phi(x)/x$ is decreasing for $x > 0$ and, in addition, that $\Phi(x)/x \to 0$ and $\Phi(x)/(\log x)^2 \to \infty$ as $x \to \infty$.

In 1985 Maier [17] showed that Selberg's result does not apply for all sufficiently large x. He proved that if λ is a real number larger than one and $\Phi(x) = (\log x)^{\lambda}$, then

$$\limsup_{x\to\infty} \frac{\pi\bigl(x + \Phi(x)\bigr) - \pi(x)}{\Phi(x)/\log x} > 1 \tag{1.5}$$

and

$$\liminf_{x\to\infty} \frac{\pi\bigl(x + \Phi(x)\bigr) - \pi(x)}{\Phi(x)/\log x} < 1. \tag{1.6}$$

The purpose of this note is to give a result which interpolates between Rankin's result (1.3) and Maier's result (1.6). In particular we shall study the behaviour of

$$\frac{\pi\bigl(x + \Phi(x)\bigr) - \pi(x)}{\Phi(x)/\log x}$$

when $\Phi(x) = (\log x)^{1+s(x)}$ where $s(x)$ is a non-increasing function of x. We shall suppose that $s(x)$ does not decrease too rapidly, in fact that

$$s(x)^{-1} = O\left(\frac{\log_2 x}{\log_4 x}\right). \tag{1.7}$$

As well, we shall suppose that

$$s(x) - s(2x) = o\left(\frac{1}{\log_2 x}\right) \tag{1.8}$$

and

$$s(x) - s(x^{3/2}) = o\big((s(x))^{3/2}\big), \tag{1.9}$$

so that $s(x)$ is a smoothly varying function of x.

In order to state our main result we require functions introduced by Dickman and Buchstab. Dickman's function $\rho(u)$ is defined [9], [5] for non-negative real numbers u as the unique continuous solution of the differential-difference equation,

$$\rho(u) = 1 \quad \text{if } 0 \leqq u \leqq 1, \tag{1.10}$$

and

$$u\rho'(u) = -\rho(u-1) \quad \text{if } u > 1. \tag{1.11}$$

Buchstab's function $\omega(u)$ is defined [1] for real numbers u which are greater than or equal to 1 as the unique continuous function for which

$$\omega(u) = \frac{1}{u} \quad \text{if } 1 \leqq u \leqq 2, \tag{1.12}$$

and

$$\big(u\omega(u)\big)' = \omega(u-1) \quad \text{if } u > 2. \tag{1.13}$$

Let γ $(= .5772\ldots)$ denote Euler's constant. Put

$$f(u,v) = v\big(\log(1+u) + \rho\big(v(1+u)\big)\big), \tag{1.14}$$

for $u > 0$, $v > 0$. It can be shown, see §2, that there is a unique positive real number θ for which

$$\min_{v \geqq 1} f(\theta, v) = \frac{e^{\gamma}}{2},$$

and that $\theta = .500462161\ldots$. We define g on the non-negative real numbers by

$$g(y) = \begin{cases} \inf\limits_{v \geqq 1} f(y,v) & \text{for } y < \theta, \\ \inf\limits_{u \geqq y} e^{\gamma}\omega(1+u) & \text{for } y \geqq \theta. \end{cases}$$

Both infima in the definition of g are minima, see §2.

We are now able to state our main result.

Theorem. *Let ε be a positive real number and let s be a non-increasing function on the positive real numbers satisfying* (1.7), (1.8) *and* (1.9). *There are arbitrarily large integers x for which*

$$\pi\big(x + (\log x)^{1+s(x)}\big) - \pi(x) < (1+\varepsilon)g\big(s(x)\big)(\log x)^{s(x)}. \tag{1.15}$$

Let λ be a positive real number and take $s(x) = \lambda$ for all positive real numbers x, so that (1.7), (1.8) and (1.9) apply. Since $\inf_{v \geqq 1} f(y, v)$ increases with y and since $\inf_{u>0} \omega(1+u) = \omega(2) = 1/2$, it follows from the definition of θ that

$$g(\lambda) \leqq \inf_{u \geqq \lambda} e^{\gamma} \omega(1+u).$$

Iwaniec [14] (see also [17]) proved that $\omega(1+u) - e^{-\gamma}$ changes sign in every interval of length 1 and so $g(\lambda) < 1$ and we recover (1.6). Further it follows from Corollary 3.2 and (3.2) of [11] that

$$g(\lambda) = 1 - \exp\bigl(-\lambda \log \lambda - \lambda \log\log \lambda + O(\lambda)\bigr).$$

In fact, if $\lambda \geqq \theta$, estimate (1.15) is already implicit in [17], see [3], and our argument coincides with that given in [17]. In particular, we shall estimate the number of primes in arithmetical progressions with moduli consisting of the product of an initial segment of the primes and then apply an averaging argument. The range when $\lambda < \theta$ requires a new approach however. In this case the moduli that we consider are the product of a segment of primes of intermediate size. Instead of exploiting the oscillatory behaviour of the Buchstab function, as we do for $\lambda \geqq \theta$, we shall make use of the fact that an initial segment of the integers can be very efficiently sieved by primes of intermediate size.

We remark, see (2.8) from §2, that

$$\lim_{y \to 0} g(y) \left(\frac{y \log(1/y)}{\log\log(1/y)} \right)^{-1} = 1. \tag{1.16}$$

Accordingly, we are able to deduce the following consequence of our main theorem.

Corollary. *Let ε be a positive real number and let s be a non-increasing function on the positive real numbers satisfying* (1.7), (1.8), (1.9) *and*

$$\lim_{x \to \infty} s(x) = 0.$$

There are arbitrarily large integers x for which

$$\pi\bigl(x + (\log x)^{1+s(x)}\bigr) - \pi(x) < (1+\varepsilon) \frac{s(x) \log\bigl(1/s(x)\bigr)}{\log\log\bigl(1/s(x)\bigr)} (\log x)^{s(x)}. \tag{1.17}$$

In particular, if we take $s(x) = \log_3 x / \log_2 x$ we see, from (1.17), that for each $\varepsilon > 0$ and for infinitely many integers x, the interval of length $\log x \log\log x$ starting at x contains at most $(1+\varepsilon)(\log_3 x)^2 / \log_4 x$ primes. Thus the average gap between primes, with the gap in or partly in the interval, is at least

$$\frac{1}{1+\varepsilon} \frac{\log x \log_2 x \log_4 x}{(\log_3 x)^2},$$

which corresponds, up to a constant factor, to Rankin's bound (1.3).

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2. Properties of the function g

In this section we shall establish properties of the functions f and g. In particular, we shall prove that θ is well defined and we shall show how to compute it and how to evaluate g. For the convenience of the reader we have included a table of values of $g(\lambda)$ for a selection of points λ from the interval $(0, 1/2)$. Furthermore we shall establish (1.16) and so show how $g(y)$ decays as y tends to 0 from above.

It follows from (1.10) and (1.11) that

$$\rho(t) = 1 - \log t \quad \text{for } 1 \leqq t \leqq 2 \tag{2.1}$$

and

$$\rho(t) = 1 - \log t + \frac{1}{2}\log^2 t + \mathrm{Li}_2\left(\frac{1}{t}\right) - \mathrm{Li}_2\left(\frac{1}{2}\right) - \frac{1}{2}\log^2 2 \quad \text{for } 2 \leqq t \leqq 3, \tag{2.2}$$

where $\mathrm{Li}_2(x)$ denotes the Euler dilogarithm function which is defined for $|x| \leqq 1$ by

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

For an explicit representation of $\rho(t)$ in the interval $3 \leqq t \leqq 4$ see the appendix of [2]. Also note that

$$\mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{1}{2}\log^2 2 = -\mathrm{Li}_2(-1) = \frac{\pi^2}{12}, \tag{2.3}$$

see (1.6) and (1.7) of [15].

By (1.11), for $v(1+u) > 1$,

$$\frac{\partial f(u,v)}{\partial v} = \log(1+u) + \rho\big(v(1+u)\big) - \rho\big(v(1+u) - 1\big).$$

Further, from (1.10), (1.14) and (2.1), for $1 < v(1+u) \leqq 2$,

$$\frac{\partial f(u,v)}{\partial v} = -\log v. \tag{2.4}$$

Furthermore, for $v(1+u) > 2$,

$$\frac{\partial^2 f(u,v)}{\partial v^2} = (1+u)\rho'\big(v(1+u)\big) - (1+u)\rho'\big(v(1+u)-1\big)$$
$$= \left(\frac{1}{v - \dfrac{1}{1+u}}\right)\rho\big(v(1+u)-2\big) - \frac{1}{v}\rho\big(v(1+u)-1\big).$$

Thus, since ρ is a non-increasing function,

$$\frac{\partial^2 f(u,v)}{\partial v^2} > 0, \tag{2.5}$$

for $v(1+u) > 2$. It follows from (2.4) and (2.5) that for each $u > 0$ there is a unique minimum of $f(u,v)$ with $v \geqq 1$. Let $v_{\min}$ $(= v_{\min}(u))$ denote the unique real number larger than 1 at which the minimum occurs. Then

$$\log(1+u) + \rho\big(v_{\min}(1+u)\big) - \rho\big(v_{\min}(1+u)-1\big) = 0. \tag{2.6}$$

In 1951 de Bruijn [7] proved that

$$\rho(t) = \exp\big(-t\big(\log t + \log_2 t + O(1)\big)\big)$$

and therefore by (2.6), since $\log(1+u) \sim u$ as $u \to 0$,

$$v_{\min}(u) = \big(1+o(1)\big)\frac{\log(1/u)}{\log\log(1/u)} \tag{2.7}$$

as $u \to 0$. Furthermore,

$$g(u) = \big(1+o(1)\big)\frac{u\log(1/u)}{\log\log(1/u)} \tag{2.8}$$

as $u \to 0$.

Put

$$h(t) = \rho(t-1) - \rho(t) \quad \text{for } t \geqq 1.$$

$\rho(t)$ is concave for $t \geqq 2$ (see [28], Lemma 3), and thus $h(t)$ is strictly decreasing for $t \geqq 2$. Therefore $h(2) = \log 2$ is the maximum of $h(t)$ for $t \geqq 2$. Thus, from (2.6), $v_{\min}(u)$ is monotone decreasing on $(0,1]$ with $v_{\min}(1) = 1$. We remark that $f(1,1) = 1$.

As we have noted above, for $u > 0$,

$$\inf_{v \geqq 1} f(u,v) = \min_{v \geqq 1} f(u,v) = f(u, v_{\min}).$$

Further, $\omega(1+u)$ is continuous and tends to $e^{-\gamma}$ as $u \to \infty$. As well, [17], $\omega(1+u) - e^{-\gamma}$ changes sign in every interval of length 1. Thus in the definition of g we may replace the two infima by minima.

Note, by (1.11), that for $u > 0$, $v \geqq 1$,

$$\frac{\partial f(u,v)}{\partial u} = \frac{v}{1+u}\big(1 - \rho\big(v(1+u) - 1\big)\big) > 0$$

and so $f\big(u, v_{\min}(u)\big)$ is an increasing continuous function of u. Therefore there is a unique positive real number θ for which

$$f\big(\theta, v_{\min}(\theta)\big) = e^{\gamma}/2. \tag{2.9}$$

We shall now evaluate θ and $v_{\min}(\theta)$. To that end observe that since $h(t)$ is strictly decreasing for $t \geqq 2$ and $h(2) = \log 2$, for each real number t with $t \geqq 2$ there is a unique real number u with

$$h(t) = \log(1+u) \tag{2.10}$$

and $0 < u \leqq 1$. Then, by (2.6),

$$v_{\min}(u) = \frac{t}{(1+u)}.$$

Notice that $f\big(u, v_{\min}(u)\big) = v_{\min}\rho\big((v_{\min})(1+u) - 1\big)$ so

$$f\big(u, v_{\min}(u)\big) = \frac{t}{1+u}\rho(t-1). \tag{2.11}$$

Taking $t = 2.7$ we find that $f\big(u, v_{\min}(u)\big) < e^{\gamma}/2$ whereas with $t = 2.6$ we have $f\big(u, v_{\min}(u)\big) > e^{\gamma}/2$. Thus θ and $v_{\min}(\theta)$ are determined by a real number t_0 with $2.6 < t_0 < 2.7$. By (2.1), (2.2) and (2.3),

$$h(t_0) = \log\left(\frac{t_0}{t_0 - 1}\right) - \left(\frac{1}{2}\log^2 t_0 + \mathrm{Li}_2\left(\frac{1}{t_0}\right) - \frac{\pi^2}{12}\right) \tag{2.12}$$

and, by (2.1), (2.9) and (2.11),

$$\log t_0 - \log(1+\theta) + \log\big(1 - \log(t_0 - 1)\big) = \log(e^{\gamma}/2). \tag{2.13}$$

Thus, by (2.10), (2.12) and (2.13),

$$\log(t_0 - 1) + \log\big(1 - \log(t_0 - 1)\big) + \frac{1}{2}\log^2 t_0 + \mathrm{Li}_2\left(\frac{1}{t_0}\right) = \log\left(\frac{e^{\gamma}}{2}\right) + \frac{\pi^2}{12}.$$

Using MAPLE we find that $t_0 = 2.637994987\ldots$ hence that

$$\theta = .500462161\ldots \tag{2.14}$$

and

$$v_{\min}(\theta) = 1.758121634\ldots. \tag{2.15}$$

For $t \geqq t_0$ we may calculate u and $f\big(u, v_{\min}(u)\big)$ from (2.10) and (2.11) and in this case $g(u) = f\big(u, v_{\min}(u)\big)$. In order to make this calculation we need to be able to evaluate $\rho(t)$ and $\rho(t-1)$ and van de Lune and Wattel [28] have described an efficient method for computing $\rho(t)$. In the table below we list values of u and $g(u)$ obtained on taking $t = 2.6 + (k/10)$ for $k = 1, \dots, 14$; here we have used the values for $\rho(t)$ given in Table II of [2].

t	λ	$g(\lambda)$	t	λ	$g(\lambda)$
2.7	.4622...	.8667...	3.4	.1452...	.4631...
2.8	.4032...	.8224...	3.5	.1208...	.4069...
2.9	.3475...	.7707...	3.6	.1000...	.3543...
3.0	.2946...	.7110...	3.7	.0824...	.3056...
3.1	.2474...	.6471...	3.8	.0675...	.2612...
3.2	.2076...	.5839...	3.9	.0550...	.2213...
3.3	.1739...	.5223...	4.0	.0446...	.1861...

Table I

In order to determine $g(\lambda)$ for $\lambda \geqq \theta$ we need to evaluate the Buchstab function numerically and Marsaglia, Zaman and Marsaglia [19] and Cheer and Goldston [3] have given efficient algorithms for this purpose. In fact Cheer and Goldston, motivated by the work of Maier [17], have determined (see [3], Table I) the initial relative maxima and minima of ω. The absolute minimum of $\omega(1+u)$ occurs at $u = 1$ and $\omega(2) = 1/2$ hence, for $\theta \leqq \lambda \leqq 1$, $g(\lambda) = e^{\gamma}/2 = .8905\dots$. By [3], Theorem 1 and Table I, the second relative minimum of $\omega(1+u)$ for $u > 0$ occurs at $\theta_2 = 2.46974\dots$ and $e^{\gamma}\omega(1+\theta_2) = .9988\dots$. From (1.12) and (1.13) we see that $\omega(1+u) = (\log u + 1)/(u+1)$ for $1 \leqq u \leqq 2$. Let θ_1 denote the real number with $1 \leqq \theta_1 \leqq 2$ for which

$$\frac{\log \theta_1 + 1}{\theta_1 + 1} = \omega(1 + \theta_2),$$

so $\theta_1 = 1.4697\dots$. Since $\omega'(1+u)$ is continuous for $u > 1$ and $\omega(1+u)$ has only one critical point, which is a local maximum, in $(1, \theta_2)$, we deduce that

$$g(\lambda) = \frac{e^{\gamma}(\log \lambda + 1)}{\lambda + 1}$$

for $1 \leqq \lambda \leqq \theta_1$ and $g(\lambda) = e^{\gamma}\omega(1+\theta_2)$ for $\theta_1 \leqq \lambda \leqq \theta_2$.

3. Preliminary lemmas

Let C be a positive real number. We say that an integer $q > 1$ is a good modulus with respect to C if $L(s, \chi) \neq 0$ for all characters $\chi \bmod q$ and all $s = \sigma + it$ with

$$\sigma > 1 - \frac{C}{\log|q(|t|+1)|}.$$

Lemma 3.1. *Let C be a positive real number and let q be an integer, with* $q > 1$, *which is a good modulus with respect to C. There exists a positive number* D_0, *which depends on C and a positive absolute constant c such that if D is a positive number with* $\log q \geqq D \geqq D_0$ *and x and h are positive integers with* $x \geqq q^D$ *and* $x/2 \leqq h \leqq x$ *then*

$$\pi(x+h,q,a) - \pi(x,q,a) = \frac{1}{\varphi(q)}\big(\mathrm{Li}(x+h) - \mathrm{Li}(x)\big)$$
$$\big(1 + O(e^{-cD} + e^{-\sqrt{\log x}})\big),$$

where the constant implied in the $O(\dots)$ *term depends on C only.*

Proof. This is [17], Lemma 2, and was deduced by Maier from work of Gallagher [12]. □

For positive real numbers x and y, let $\psi(x, y)$ denote the number of positive integers n with n at most x for which the greatest prime factor of n is at most y.

Lemma 3.2. *Let* ε *be a positive real number, let x and y be real numbers and put* $u = (\log x)/\log y$. *If*

$$u \leqq (\log x)^{\frac{3}{8}-\varepsilon}$$

then

$$\psi(x, y) = x\rho(u)\big(1 + o(1)\big),$$

as $x \to \infty$.

Proof. This is [20], Lemma 3.20, see also [8], (1.4). □

For positive real numbers x and y, let $\phi(x, y)$ denote the number of positive integers n with n at most x and with all prime factors of n at least y. Further, note that the letter p under a product sign indicates that the product is taken over prime numbers only.

Lemma 3.3. *Let x and y be real numbers and put* $u = (\log x)/\log y$. *If u is fixed and* $u > 1$, *then*

$$\phi(x, y) = xe^{\gamma}\omega(u)\prod_{p\leqq y}\left(1 - \frac{1}{p}\right)\big(1 + o(1)\big)$$

as $x \to \infty$.

Proof. This follows from [1] and Mertens' Theorem, see also [5]. □

4. Proof of main theorem

Let ε be a real number with $0 < \varepsilon < 1$. We shall denote by δ a real number with $0 < \delta < 1$ which depends on ε and by D a positive integer which depends on δ.

As before let θ be the positive real number for which $g(\theta) = e^{\gamma}/2$. Put $\beta = \lim_{x \to \infty} s(x)$; β exists since $s(x)$ is non-increasing. We distinguish two cases. In the first case $\beta < \theta$ while in the second case $\beta \geqq \theta$.

For each positive integer z put

$$P_1(z) = \prod_{p \leqq z} p$$

and

$$\Delta_1(z) = 2P_1(z)^D.$$

If $\beta < \theta$ find $v_0 \geqq 1$ such that $f\big(s(\Delta_1(z)), v\big)$ is minimized at v_0. We may suppose by taking z sufficiently large, that $v_0 \geqq 1.7$, see (2.15). Put

$$P(z) = \begin{cases} \prod_{z^{1/v_0} \leqq p \leqq z} p & \text{if } \beta < \theta, \\ P_1(z) & \text{if } \beta \geqq \theta, \end{cases}$$

and

$$\Delta(z) = P(z)^D. \tag{4.1}$$

Notice that since $v_0^{-1} < 3/5$,

$$\log P(z) = \big(1 + o(1)\big)z, \tag{4.2}$$

by the prime number theorem.

In 1935 Page [21], see also [4], p. 95, proved there is a positive number C_0 such that there is at most one primitive character χ modulo q for an integer $q \leqq P(z)$ for which $L(s, \chi)$ has a zero s with $s = \sigma + it$ and

$$\sigma > 1 - \frac{C_0}{\log|q(|t| + 1)|}. \tag{4.3}$$

Further if such a zero exists it is real, unique and is associated with a real character. If no such zero exists, then certainly $L(s, \chi)$ is non-zero for

$$\mathrm{Re}(s) > 1 - \frac{C_0}{\log\big(P(z)(|t| + 1)\big)}, \tag{4.4}$$

and for all characters χ modulo $P(z)$. If such a zero σ exists, then we may choose z' such that

$$1 - \frac{C_0}{2\log P(z')} > \sigma > 1 - \frac{C_0}{\log P(z')}.$$

In this case $L(s, \chi)$ is non-zero for

$$\mathrm{Re}(s) > 1 - \frac{C_0}{2\log\big(P(z')(|t|+1)\big)}.$$

Therefore there exist arbitrarily large integers z for which $L(s, \chi)$ is non-zero for

$$\mathrm{Re}(s) > 1 - \frac{C_0}{2\log\big(P(z)(|t|+1)\big)},$$

and for all characters χ modulo $P(z)$. We shall assume henceforth that z is such an integer. Thus $P(z)$ is a "good" modulus with respect to the constant $C_0/2$ in the sense introduced by Maier in [16].

Suppose that $\beta \geqq \theta$ and let λ be the smallest real number with $\lambda \geqq \beta$ for which

$$\omega(1+\lambda) = \min_{u \geqq \beta} \omega(1+u).$$

If $\lambda > \beta$, choose δ to satisfy

$$(4.5) \qquad (1+\delta)\beta < \lambda$$

and put

$$(4.6) \qquad U = \left[\big((1+\delta)zD\big)^{1+\lambda}\right].$$

If $\lambda = \beta$ or if $\beta < \theta$ put

$$(4.7) \qquad U = \left[\big((1+\delta)zD\big)^{1+s(\Delta(z))}\right].$$

Let $\mathscr{R}$ denote the set of integers from 1 to U which are coprime with $P(z)$. Let S denote the number of primes of the form $P(z)k + \ell$ with $1 \leqq \ell \leqq U$ and $P(z)^{D-1} < k \leqq 2P(z)^{D-1}$. For each integer ℓ with $1 \leqq \ell \leqq U$ which is coprime with $P(z)$ we may estimate the number of primes of the form $P(z)k + \ell$ with $P(z)^{D-1} < k \leqq 2P(z)^{D-1}$ by Lemma 3.1 with $a = \ell$, $q = P(z)$ and $x = h = P(z)^D$. By (4.2), for z sufficiently large, the number is at most

$$\frac{1}{\varphi\big(P(z)\big)} \frac{P(z)^D}{\log\big(P(z)^D\big)} \big(1 + O(e^{-cD})\big)$$

hence, for D sufficiently large in terms of δ, at most

$$(1+\delta)\frac{P(z)^{D-1}}{\log\big(P(z)^D\big)} \prod_{p \mid P(z)} \left(1 - \frac{1}{p}\right)^{-1}.$$

Therefore

$$S \leqq (1+\delta)|\mathscr{R}| \frac{P(z)^{D-1}}{\log\big(P(z)^D\big)} \prod_{p \mid P(z)} \left(1-\frac{1}{p}\right)^{-1}.$$

Thus for some k with $P(z)^{D-1} < k \leqq 2P(z)^{D-1}$ the number of primes in the interval $[P(z)k+1, P(z)k+U]$ is at most

$$(4.8) \qquad (1+\delta) \frac{|\mathscr{R}|}{\log\big(P(z)^D\big)} \prod_{p \mid P(z)} \left(1-\frac{1}{p}\right)^{-1}.$$

We shall now estimate $|\mathscr{R}|$. Suppose first that $\beta < \theta$. Put

$$\mathscr{R}_1 = \{1 \leqq n \leqq U\colon \text{the greatest prime factor of } n \text{ is less than } z^{1/v_0}\}$$

and

$$\mathscr{R}_2 = \{1 \leqq n \leqq U\colon n \text{ is divisible by a prime } p \text{ with } p > z\}.$$

Note that

$$(4.9) \qquad \mathscr{R} \subseteqq \mathscr{R}_1 \cup \mathscr{R}_2.$$

Observe that, for z sufficiently large,

$$(4.10) \qquad \begin{aligned} \frac{\log U}{v_0^{-1}\log z} &= v_0\big(1+s\big(\Delta(z)\big)\big)\left(1+\frac{\log(1+\delta)D}{\log z}+O\left(\frac{1}{z}\right)\right) \\ &\geqq v_0\big(1+s\big(\Delta(z)\big)\big). \end{aligned}$$

If $\beta > 0$ then $v_0\big(1+s\big(\Delta(z)\big)\big)$ is bounded as z tends to infinity. If $\beta = 0$ then, by (2.7),

$$v_0 = \big(1+o(1)\big) \frac{\log\left(\dfrac{1}{s\big(\Delta_1(z)\big)}\right)}{\log\log\left(\dfrac{1}{s\big(\Delta_1(z)\big)}\right)}.$$

Since $s(x) > 1/\log\log x$ for x sufficiently large by (1.7) and $\Delta_1(z) < e^{2zD}$, for z sufficiently large, we find that

$$s\big(\Delta_1(z)\big) > s(e^{2zD}) > \frac{1}{\log 2zD},$$

hence that

$$(4.11) \qquad v_0 = O(\log\log z).$$

In particular, by (4.7) and (4.11),

$$\frac{\log U}{v_0^{-1}\log z} = O\big((\log U)^{1/4}\big)$$

whence, by Lemma 3.2,

$$|\mathscr{R}_1| = U\rho\left(\frac{\log U}{v_0^{-1}\log z}\right)\big(1+o(1)\big).$$

Since ρ is a non-increasing function of u

$$|\mathscr{R}_1| \leqq U\rho\Big(v_0\big(1+s\big(\Delta(z)\big)\big)\Big)\big(1+o(1)\big). \tag{4.12}$$

There exists a positive real number B_1 such that

$$\sum_{p\leqq z}\frac{1}{p} = \log\log z + B_1 + O\left(\frac{1}{\log z}\right), \tag{4.13}$$

see [13], 22.7.4. Thus by (4.7) and (4.13),

$$\begin{aligned}|\mathscr{R}_2| &\leqq U\sum_{z<p\leqq U}\frac{1}{p} = U\left(\log\log U - \log\log z + O\left(\frac{1}{\log z}\right)\right)\\ &= U\left(\log\big(1+s\big(\Delta(z)\big)\big) + \log\left(\log z\left(1+\frac{\log(1+\delta)D}{\log z}\right)\right) - \log\log z + O\left(\frac{1}{\log z}\right)\right).\end{aligned}$$

But by (4.1) and (4.2), $\Delta(z) < e^{2zD}$ for z sufficiently large and so, by (1.7),

$$|\mathscr{R}_2| \leqq U\Big(\log\big(1+s\big(\Delta(z)\big)\big)\big(1+o(1)\big)\Big). \tag{4.14}$$

Therefore, by (4.9), (4.12), and (4.14),

$$|\mathscr{R}| \leqq U\Big(\rho\Big(v_0\big(1+s\big(\Delta(z)\big)\big)\Big) + \log\big(1+s\big(\Delta(z)\big)\big)\big(1+o(1)\big)\Big).$$

Thus from (4.8) and Mertens' Theorem, see [13], Theorem 429, the number of primes in the interval $[P(z)k+1, P(z)k+U]$ is at most

$$(1+2\delta)\frac{U}{\log\big(\Delta(z)\big)}\Big(v_0\Big(\log\big(1+s\big(\Delta(z)\big)\big) + \rho\Big(v_0\big(1+s\big(\Delta(z)\big)\big)\Big)\Big)\Big). \tag{4.15}$$

Take $x = P(z)k$. Then

$$\Delta(z) < x \leqq 2\Delta(z). \tag{4.16}$$

Further, from (4.1) and (4.2),

$$\log\Delta(z) = \big(1+o(1)\big)zD. \tag{4.17}$$

Thus, for z sufficiently large,

$$U > \left(\left(1+\frac{\delta}{2}\right)zD\right)^{1+s(\Delta(z))} > \big(\log\big(2\Delta(z)\big)\big)^{1+s(\Delta(z))},$$

and, since s is non-increasing,

$$U > (\log x)^{1+s(x)}. \tag{4.18}$$

On the other hand, by (4.7) and (4.17),

$$\begin{aligned}\frac{U}{\log\big(\Delta(z)\big)} &\leqq (1+2\delta)(zD)^{s(\Delta(z))}\\ &\leqq (1+2\delta)\big((1+\delta)\log x\big)^{s(\Delta(z))}\\ &\leqq (1+2\delta)(1+\delta)^{s(1)}(\log x)^{s(\Delta(z))}\end{aligned}$$

and so, by (1.8) and (4.16),

$$\frac{U}{\log\big(\Delta(z)\big)} \leqq (1+2\delta)(1+\delta)^{s(1)}(\log x)^{o\left(\frac{1}{\log\log x}\right)}(\log x)^{s(x)}.$$

Thus we can choose δ sufficiently small and z sufficiently large so that

$$\frac{U}{\log\big(\Delta(z)\big)} \leqq \left(1+\frac{\varepsilon}{2}\right)(\log x)^{s(x)}. \tag{4.19}$$

Since $x \leqq \Delta_1(z)$ and s is non-increasing, $s(x) \geqq s\big(\Delta_1(z)\big)$ hence

$$g\big(s(x)\big) \geqq g\big(s\big(\Delta_1(z)\big)\big). \tag{4.20}$$

Suppose that $\beta > 0$. Then $g(\beta) > 0$. Further $f(u, v_0)$ is a continuous function of u for u positive, $\Delta(z) \leqq \Delta_1(z)$ and $\lim_{z\to\infty} s\big(\Delta(z)\big) = \beta$. Thus, for z sufficiently large,

$$f\big(s\big(\Delta(z)\big), v_0\big) < (1+\delta)f\big(s\big(\Delta_1(z)\big), v_0\big) = (1+\delta)g\big(s\big(\Delta_1(z)\big)\big). \tag{4.21}$$

Suppose next that $\beta = \lim_{z\to 0} s\big(\Delta(z)\big) = 0$. Then, since $s\big(\Delta(z)\big) \geqq s\big(\Delta_1(z)\big)$ and $\rho(u)$ is a non-increasing function of u for u positive,

$$v_0\rho\big(v_0\big(1+s\big(\Delta(z)\big)\big)\big) \leqq v_0\rho\big(v_0\big(1+s\big(\Delta_1(z)\big)\big)\big). \tag{4.22}$$

Further, by the prime number theorem,

$$\frac{\Delta_1(z)}{\Delta(z)} \leqq 2\Bigg(\prod_{p\leqq z^{1/v_0}} p\Bigg)^{D} = e^{(1+o(1))z^{1/v_0}D}.$$

Since $\Delta(z) \leqq \Delta_1(z)$, $v_0^{-1} \leqq 3/4$ and (4.17) holds, for z sufficiently large,

$$\Delta(z) \leqq \Delta_1(z) \leqq \Delta(z)^{3/2}. \tag{4.23}$$

Therefore, by (4.23), (1.9) and the fact that $\beta = 0$,

$$s(\Delta(z)) < \left(1 + \frac{\delta}{2}\right) s(\Delta_1(z))$$

and thus

$$v_0 \log(1 + s(\Delta(z))) < (1 + \delta) v_0 \log(1 + s(\Delta_1(z))), \tag{4.24}$$

for z sufficiently large. It follows from (4.22) and (4.24) that (4.21) holds when $\beta = 0$ and when $0 < \beta < \theta$.

Therefore, from (4.15), (4.19), (4.20) and (4.21), provided that δ is sufficiently small that $(1 + 2\delta)(1 + \delta)(1 + \varepsilon/2) < 1 + \varepsilon$ we find that the interval starting at x of length $(\log x)^{1+s(x)}$ contains at most

$$(1 + \varepsilon) g(s(x)) (\log x)^{s(x)}$$

prime numbers as required.

Suppose now that $\beta \geqq \theta$. It follows from Lemma 3.3 that, for z sufficiently large,

$$|\mathscr{R}| \leqq (1 + \delta) U \prod_{p \mid P(z)} \left(1 - \frac{1}{p}\right) e^{\gamma} \omega(1 + \lambda)$$

hence, by (4.8), the number of primes in the interval $[P(z)k + 1, P(z)k + U]$ is at most

$$(1 + \delta)^2 \frac{U}{\log(P(z)^D)} e^{\gamma} \omega(1 + \lambda). \tag{4.25}$$

If $\lambda = \beta$ we take $x = P(z)k$ so that $\Delta(z) < x \leqq 2\Delta(z)$. Then, by (4.1), (4.2) and (4.7),

$$U > (\log x)^{1+s(x)}$$

and (4.19) holds, as before, for z sufficiently large. Therefore, the number of primes in the interval $[x, x + U]$ is at most

$$(1 + \delta)^2 \left(1 + \frac{\varepsilon}{2}\right) e^{\gamma} \omega(1 + \beta) (\log x)^{s(x)},$$

which, since ω, hence g, is continuous and $\lim_{x \to \infty} s(x) = \beta$, is at most

$$(1 + \delta)^3 \left(1 + \frac{\varepsilon}{2}\right) g(s(x)) (\log x)^{s(x)},$$

for z sufficiently large. Choosing δ so that $(1 + \delta)^3 (1 + \varepsilon/2) < 1 + \varepsilon$ our result follows.

On the other hand if $\lambda > \beta$ we put $\ell = \left[\left((1+\delta)zD\right)^{1+s(\Delta(z))}\right]$. Then, by (4.25), there is a subinterval of $[P(z)k+1, P(z)k+U]$ of length ℓ with at most

$$(1+\delta)^3 \frac{\ell}{\log\left(P(z)^D\right)} e^{\gamma}\omega(1+\lambda)$$

primes. Take x to be the start of that subinterval. Since $x \leqq 2\Delta(z) + U \leqq 3\Delta(z)$ for z sufficiently large, by (4.1) and (4.2),

$$\ell > (\log x)^{1+s(x)} \tag{4.26}$$

and the number of primes in the interval $[x, x+\ell]$ is at most

$$(1+\delta)^4 \left(1+\frac{\varepsilon}{2}\right) e^{\gamma}\omega(1+\lambda)(\log x)^{s(x)}, \tag{4.27}$$

as in the proof of (4.19). For z sufficiently large,

$$\beta \leqq s(x) \leqq (1+\delta)\beta,$$

hence, by (4.5),

$$g\left(s(x)\right) = e^{\gamma}\omega(1+\lambda). \tag{4.28}$$

On choosing δ so that $(1+\delta)^4(1+\varepsilon/2) < 1+\varepsilon$ our result follows from (4.26), (4.27) and (4.28). □

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Abteilung für Mathematik, Universität Ulm, Oberer Eselsberg, 89081 Ulm, Germany
e-mail: hamaier@mathematik.uni-ulm.de

Department of Pure Mathematics, The University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
e-mail: cstewart@uwaterloo.ca

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