Math. Proc. Camb. Phil. Soc. (1976), 79, 483
MPCPS 79–42
Printed in Great Britain

Operations on K-theory of torsion-free spaces

BY J. F. ADAMS AND P. HOFFMAN
University of Cambridge, and University of Waterloo, Ontario

(*Received* 5 *June* 1975)

1. *Introduction.* The object of this paper is to study stable operations on the K-cohomology of torsion-free spaces and spectra. Such a study was begun by Novikov ((4), §8). We hope the present work may among other things serve to clarify some of the obscure points in this part of his work.

To proceed to the details, let K be the BU-spectrum, and let K_n, K^n be the corresponding homology and cohomology functors; we need them only for $n = 0$. Let n, m be integers, and let $C(n, m)$ be the full subcategory of CW-spectra X which satisfy the following conditions:

(i) $\pi_r(X) = 0$ for $r < 2n$.

(ii) $H_r(X)$ is free for all r.

(iii) $H_r(X) = 0$ for $r > 2m$.

Let $A(n, m)$ be the algebra of operations

$$a: K^0(X) \to K^0(X)$$

which are defined and natural for X in $C(n, m)$. The object of this paper is to calculate $A(n, m)$.

2. *Results.* In this section we will explain the theory and its results; the proofs will mostly be postponed to the next section.

We propose to describe $A(n, m)$ by duality. The first step is to introduce an object $A_0(n, m)$ such that $A(n, m)$ will turn out to be the dual of $A_0(n, m)$. The subscript zero may be interpreted as 'degree zero'; it would be inappropriate to write $A_*(n, m)$ because no other degree is to be considered.

Let X be a spectrum in $C(n, m)$, let x be an element of $K_0(X)$, and let $f: X \to K$ be a map. Then we can form the element $f_* x$ in $K_0(K)$; we define $A_0(n, m)$ to be the subset of $K_0(K)$ consisting of all such elements $f_* x$, as X, x and f run over all possibilities.

PROPOSITION 2·1. (*a*) *$A_0(n, m)$ is a finitely-generated free Z-module.* (*b*) *The product $K_0(K) \otimes K_0(K) \to K_0(K)$ maps $A_0(n, m) \otimes A_0(n', m')$ into $A_0(n + n', m + m')$.*

Here $\otimes$ means $\otimes_Z$, as usual.

We pause to prove part (*b*), because the ideas will be useful in this section. Take objects $X \in C(n, m)$, $Y \in C(n', m')$, elements $x \in K_0(X)$, $y \in K_0(Y)$, and maps $f: X \to K$, $g: Y \to K$, so that $f_* x$, $g_* y$ are typical elements of $A_0(n, m)$, $A_0(n', m')$. Then we may form the smash product $X \wedge Y$ and it lies in $C(n + n', m + m')$. We may also form the external product $xy \in K_0(X \wedge Y)$, and the map

$$X \wedge Y \xrightarrow{f \wedge g} K \wedge K \xrightarrow{\mu} K,$$

where μ is the product map for the ring-spectrum K. If we recall that f is an elemen of $K^0(X)$, and g is an element of $K^0(Y)$, then the map just constructed is their externa product $fg \in K^0(X \wedge Y)$. The product in $K_0(K)$ of $f_* x$ and $g_* y$ is $(fg)_*(xy)$, and there fore it lies in $A_0(n+n', m+m')$.

Of course, a similar proof with the wedge-sum $X \vee Y$ shows that $A_0(n,m)$ is a Z module.

Proposition 2·1 allows one to continue with the theory; examples and a precis calculation of $A_0(n,m)$ will be given below – see (2·5) to (2·9).

PROPOSITION 2·2. (*a*) *If* X *lies in* $C(n,m)$, *then the coaction map*

$$\nu : K_0(X) \to K_0(K) \otimes K_0(X)$$

maps $K_0(X)$ *into* $A_0(n,m) \otimes K_0(X)$.

(*b*) *The coproduct map* $\psi : K_0(K) \to K_0(K) \otimes K_0(K)$ *maps* $A_0(n,m)$ *into*

$$A_0(n,m) \otimes A_0(n,m).$$

Some comments on the statement are in order. According to the details given in (1 the values of the coaction map ν lie in $K_*(K) \otimes_{\pi_*(K)} K_*(X)$, and the values of the co product map ψ lie in $K_*(K) \otimes_{\pi_*(K)} K_*(K)$. However, the 0-dimensional component of these graded groups are $K_0(K) \otimes_Z K_0(X)$ and $K_0(K) \otimes_Z K_0(K)$ (up to obvious iso morphisms). Again, one should check that the maps

$$A_0(n,m) \otimes K_0(X) \to K_0(K) \otimes K_0(X)$$
$$A_0(n,m) \otimes A_0(n,m) \to K_0(K) \otimes K_0(K)$$

are monomorphic; this is immediate, because $K_0(X)$ is torsion-free for X in $C(n,m$ and $K_0(K)$ is torsion-free by (3).

If X lies in $C(n,m)$ then $K_0(X)$ is a free Z-module and we have

$$K^0(X) \cong \mathrm{Hom}_Z(K_0(X), Z),$$

the isomorphism being defined by the Kronecker product. Let $A^0(n,m)$ be

$$\mathrm{Hom}_Z(A_0(n,m), Z),$$

the dual of $A_0(n,m)$. By duality from Proposition 2·2 we obtain an action map

$$A^0(n,m) \otimes K^0(X) \to K^0(X)$$

and a product map

$$A^0(n,m) \otimes A^0(n,m) \to A^0(n,m).$$

Thus $K^0(X)$ becomes a module over the ring $A^0(n,m)$. We thus obtain a hom morphism of rings

$$\theta : A^0(n,m) \to A(n,m).$$

THEOREM 2·3. *This homomorphism* $\theta : A^0(n,m) \to A(n,m)$ *is an isomorphism.*

COROLLARY 2·4. *The ring $A(n, m)$ is commutative.*

This follows immediately, because the coproduct ψ in $K_0(K)$ is commutative (3).

In order to proceed further, we need notation for elements of $K_0(K)$. According to (3), $K_0(K)$ is torsion-free, so that we have an embedding

$$K_0(K) \to K_0(K) \otimes Q.$$

Here $K_*(K) \otimes Q$ is the ring of finite Laurent series $Q[u, v, u^{-1}, v^{-1}]$ on two generators u and v, of degree 2, which are described in (3). In particular, $K_0(K) \otimes Q$ is the ring of finite Laurent series $Q[w, w^{-1}]$, where $w = u^{-1}v$. Alternatively, we may introduce w by the method of the following example.

EXAMPLE 2·5. *Let r be an integer (positive, negative or zero) and let X be the sphere-spectrum S^{2r}; it lies in $C(r, r)$. Let x be the generator for $K_0(S^{2r}) \cong \pi_{-2r}(K) \cong Z$, and let $f: S^{2r} \to K$ be the generator for $K^0(S^{2r}) \cong \pi_{2r}(K) \cong Z$. Then $f_* x = w^r$; thus w^r lies in $A_0(r, r)$. Since the objects in $C(r, r)$ are (up to equivalence) wedge-sums of S^{2r}, $A_0(r, r)$ is the Z-module generated by w^r.*

The proof is trivial, and will be omitted.

COROLLARY 2·6. *Multiplication by w^r defines an isomorphism*

$$A_0(n, m) \xrightarrow{\cong} A_0(n+r, m+r)$$

whose inverse is multiplication by w^{-r}.

This follows immediately from (2·1b) and (2·5). Of course the result merely reflects the fact that the categories $C(n, m)$ and $C(n+r, m+r)$ are equivalent; the equivalences are given by iterated suspension, that is, multiplication by S^{2r}, S^{-2r}.

EXAMPLE 2·7. *Let ξ be the canonical line bundle over CP^m, so that $K^0(CP^m)$ has a base consisting of the powers $(\xi - 1)^i$ for $0 \leqslant i \leqslant m$; let $\{x_i\}$ be the dual base for $K_0(CP^m)$. Let X be the suspension-spectrum of CP^m, so that X lies in $C(1, m)$; we recall that $K^0(X) = \tilde{K}^0(CP^m)$, and similarly for K_0. Let $f: X \to K$ represent $\xi - 1$. Then*

$$f_* x_m = \frac{w(w-1)(w-2)\dots(w-m+1)}{1.2.3\dots m}.$$

Thus

$$\frac{w(w-1)(w-2)\dots(w-m+1)}{1.2.3\dots m} = \binom{w}{m}$$

lies in $A_0(1, m)$.

This formula is due to Adams, Harris & Switzer ((3), p. 407, Proposition 6·13); but the methods of the present paper permit a shorter and easier proof.

COROLLARY 2·8. *Let*

$$p_r(w) = \frac{(w-1)(w-2)\dots(w-r)}{2.3\dots r+1} = w^{-1}\binom{w}{r+1};$$

then $p_r(w)$ lies in $A_0(0, r)$.

This follows immediately from (2·6) and (2·7).

THEOREM 2·9. *$A_0(n, m)$ is the Z-submodule of $Q[w, w^{-1}]$ generated by the products*

$$w^n p_{r_1}(w) p_{r_2}(w) \dots p_{r_\nu}(w)$$

with $r_1 \geq 1, r_2 \geq 1, \dots, r_\nu \geq 1, r_1 + r_2 + \dots + r_\nu \leq m - n$ (and therefore with $\nu \leq m - n$).

Of course the empty product is interpreted as 1. The fact that these products do lie in $A_0(n, m)$ follows immediately from (2·1 *b*) and (2·8). This completes the statement of results.

3. *Proofs.* In this section we will prove the results stated in section 2.

First we note that while there appears to be no 'universal example' for the problem of computing $A(n, m)$, there is a splendid one for computing $A_0(0, m)$, namely the Thom spectrum MU. More precisely, we filter $K_0(\mathrm{MU})$ in the usual way, taking $K_0(\mathrm{MU})_{2m}$ to be the image of

$$K_0(\mathrm{MU}^{2m}) \to K_0(\mathrm{MU}),$$

where MU^{2m} is the $2m$-skeleton of MU.

LEMMA 3·1. *$A_0(0, m)$ is the image of $K_0(\mathrm{MU})_{2m}$ under the homomorphism*

$$t_*: K_0(\mathrm{MU}) \to K_0(K)$$

induced by the Todd map $t: \mathrm{MU} \to K$.

Proof. The skeleton MU^{2m} qualifies as an object of $C(0, m)$ (and can be chosen finite if required). Therefore the image of

$$K_0(\mathrm{MU}^{2m}) \to K_0(K)$$

is contained in $A_0(0, m)$, by definition.

Conversely, let X be an object of $C(0, m)$, and let $f: X \to K$ be a map. Since X is (-1)-connected, the map f lifts to the connective BU-spectrum k. The Atiyah–Hirzebruch spectral sequences for $[X, \mathrm{MU}]$ and $[X, k]$ are both trivial since $H_*(X)$ is free. The induced homomorphism

$$\pi_*(\mathrm{MU}) \to \pi_*(k)$$

is split epi, so that

$$H^*(X; \pi_*(\mathrm{MU})) \to H^*(X; \pi_*(k))$$

is split epi. Now an easy spectral sequence argument shows that the induced homomorphism

$$[X, \mathrm{MU}] \to [X, k]$$

is epi; so the map f lifts to MU. Since X has homological dimension $\leq 2m$, the map f lifts to MU^{2m}. It is now clear that any class $f_* x$ with $x \in K_0(X)$ is contained in the image of $K_0(\mathrm{MU}^{2m}) \to K_0(K)$. This proves Lemma 3·1.

Proof of Proposition 2·1 (*a*). By Corollary 2·6 it is sufficient to consider the case $n = 0$. In Lemma 3·1 it is clear that $K_0(\mathrm{MU})_{2m}$ is a finitely-generated Z-module; therefore its image $A_0(0, m)$ is a finitely-generated Z-module. It is torsion-free, since $K_0(K)$ is torsion-free by (3); therefore $A_0(0, m)$ is free. This proves Proposition 2·1 (*a*).

To prove Proposition 2·2 we need another lemma.

LEMMA 3·2. *Let* $\nu: K_0(X) \to K_0(K) \otimes K_0(X)$ *be the coaction map; suppose* $x \in K_0(X)$ *and* $\nu(x) = \sum_i k_i \otimes x_i$, *where* $k_i \in K_0(K)$ *and* $x_i \in K_0(X)$; *let* $f: X \to K$ *be a map. Then*

$$f_* x = \sum_i k_i \langle f, x_i \rangle \in K_0(K).$$

On the right-hand side, f is interpreted as an element of $K^0(X)$; since $x_i \in K_0(X)$, we can form the Kronecker product $\langle f, x_i \rangle$, and it lies in Z.

Proof. This is one of the basic properties of ν – see (1), p. 74. For our present purposes we may deduce it from the following commutative diagram, where ϵ is as in (1).

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\nu} & K_0(K) \otimes K_0(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle 1\otimes f_*} \\ K_0(K) & \xrightarrow{\psi} & K_0(K) \otimes K_0(K) \\ & \searrow{\scriptstyle 1} & \downarrow{\scriptstyle 1\otimes \epsilon} \\ & & K_0(K) \otimes_Z Z \end{array}$$

The composite

$$K_0(X) \xrightarrow{f_*} K_0(K) \xrightarrow{\epsilon} Z$$

carries x_i to $\langle f, x_i \rangle$. This proves Lemma 3·2.

Proof of Proposition 2·2. We start with part (a). Let X lie in $C(n, m)$. Then $K_0(X)$ is a free Z-module and $K^0(X) \cong \mathrm{Hom}_Z(K_0(X), Z)$; let us take dual bases $\{x_i\}$ in $K_0(X)$ and $\{f_i\}$ in $K^0(X)$. If $x \in K_0(X)$, we may write

$$\nu x = \sum_i k_i \otimes x_i$$

as in Lemma 3·2; and using that lemma,

$$k_j = \sum_i k_i \langle f_j, x_i \rangle = (f_j)_* x \in A_0(n, m).$$

Thus νx lies in $A_0(n, m) \otimes K_0(X)$. This proves part (a).

Part (b) follows from part (a) by naturality, since every element of $A_0(n, m)$ comes via a map $f: X \to K$. This proves Proposition 2·2.

In order to prove Theorem 2·3, we need to make the homomorphism

$$\theta: A^0(n, m) \to A(n, m)$$

more explicit.

LEMMA 3·3. *Let* $\alpha \in A^0(n, m)$; *then the corresponding operation* $a = \theta\alpha \in A(n, m)$ *is determined by the equation*

$$\langle af, x \rangle = \alpha(f_* x)$$

valid for each $X \in C(n, m)$, *each* $f: X \to K$ *and each* $x \in K_0(X)$.

Proof. Let $\nu x = \sum_i k_i \otimes x_i$ with $k_i \in A_0(n, m)$, $x_i \in K_0(X)$. Then the definition of $a = \theta\alpha$ by duality is

$$\langle af, x \rangle = \sum_i \alpha(k_i) \langle f, x_i \rangle.$$

Since α is linear we have

$$\sum_i \alpha(k_i)\langle f, x_i\rangle = \alpha(\sum_i k_i\langle f, x_i\rangle),$$

and by Lemma 3·2 we have

$$\sum_i k_i\langle f, x_i\rangle = f_* x.$$

This proves Lemma 3·3.

COROLLARY 3·4. The map θ is monomorphic.

Proof. Consider the equation

$$\langle af, x\rangle = \alpha(f_* x).$$

If $a = 0$, then α vanishes on every element $f_* x$ of $A_0(n, m)$.

To prove that θ is epimorphic requires a few preliminaries. We write KQ for the BU-spectrum with rational coefficients.

LEMMA 3·5. *Let $b, c: K^0(X) \to KQ^0(X)$ be operations, defined and natural for X in $C(n, m)$, which agree on S^{2r} for $n \leqslant r \leqslant m$. Then $b = c$.*

Proof. Let $a = b - c$; then $a: K^0(X) \to KQ^0(X)$ is an operation defined and natural for X in $C(n, m)$, and zero on S^{2r} for $n \leqslant r \leqslant m$; it will be sufficient to prove that $a = 0$. For any X in $C(n, m)$, any maps $f: X \to K$ and $g: S^{2r} \to X$, and any $s \in K_0(S^{2r})$ we have

$$\langle af, g_* s\rangle = \langle g^*(af), s\rangle = \langle a(g^* f), s\rangle = 0$$

(since a is zero on $K^0(S^{2r})$). Now

$$K_*(K) \otimes Q \cong \pi_*(K) \otimes \pi_*(X) \otimes Q;$$

therefore such classes $g_* s$ generate $K_0(X) \otimes Q$ as a Q-module. But

$$\begin{aligned} KQ^0(X) &\cong \mathrm{Hom}_Z(K_0(X), Q) \\ &\cong \mathrm{Hom}_Q(K_0(X) \otimes Q, Q); \end{aligned}$$

so we may conclude that $af = 0$. Since this holds for all X and f, we have $a = 0$. This proves Lemma 3·5.

Next we need to describe $KQ^0(KQ)$, the algebra of operations

$$a: KQ^0(X) \to KQ^0(X)$$

defined and natural for all spectra X. For any X there is an isomorphism

$$ch: KQ^0(X) \xrightarrow{\cong} \prod_{r \in Z} H^{2r}(X; Q).$$

Since Q acts on $H^{2r}(X; Q)$, the algebra $\prod_{r \in Z} Q$ acts on $\prod_{r \in Z} H^{2r}(X; Q)$. More precisely, an element of $\prod_{r \in Z} Q$ is a vector $\{\lambda_r\}$ with $\lambda_r \in Q$ for each r; an element of $\prod_{r \in Z} H^{2r}(X; Q)$ is a vector $\{h_r\}$ with $h_r \in H^{2r}(X; Q)$; and the action is given by

$$\{\lambda_r\}\{h_r\} = \{\lambda_r h_r\}.$$

This defines a map of algebras

$$\prod_{r\in Z} Q \to KQ^0(KQ);$$

it is easy to show that this map is an isomorphism, but we will not need this fact. We may write $\sum_r \lambda_r \pi_r$ for the operation corresponding to $\{\lambda_r\}$; here π_r is the idempotent operation corresponding to the projection of $\prod_{s\in Z} H^{2s}(X;Q)$ on its rth factor.

Let $j: K\to KQ$ be the obvious map.

LEMMA 3·6. *Let $a\in A(n,m)$ be an operation. Then there is a map $b: K\to KQ$ such that the composite operation*

$$K^0(X) \xrightarrow{a} K^0(X) \xrightarrow{j} KQ^0(X)$$

is induced by b for all X in $C(n,m)$. That is, for any $f: X\to K$ we have

$$j(af) = bf.$$

Proof. Let $a\in A(n,m)$ be an operation. Then for r in the range $n\leqslant r\leqslant m$, a must act on $K^0(S^{2r})$ by multiplication with a scalar $\lambda_r\in Z$. Choose scalars λ_r arbitrarily for $r<n$ and for $r>m$, and let b be the composite

$$K \xrightarrow{j} KQ \xrightarrow{\sum_r \lambda_r\pi_r} KQ.$$

Then by construction, the composite operation

$$K^0(X) \xrightarrow{a} K^0(X) \xrightarrow{j} KQ^0(X)$$

agrees with that induced by b when $X = S^{2r}$, $n\leqslant r\leqslant m$; so the two operations are equal by Lemma 3·5. This proves Lemma 3·6.

Proof of Theorem 2·3. After (3·4), it remains to prove that θ is epimorphic. Let $a\in A(n,m)$ be an operation, and let $b: K\to KQ$ be as in Lemma 3·6. We can define a linear function

$$\alpha: A_0(n,m)\to Q$$

by

$$\alpha(f_* x) = \langle b, f_* x\rangle;$$

then α maps $A_0(n,m)$ into Z, and $\theta\alpha = a$. In fact, for any $X\in C(n,m)$, any $f: X\to K$ and any $x\in K_0(X)$ we have

$$\langle b, f_* x\rangle = \langle f^* b, x\rangle = \langle bf, x\rangle = \langle j(af), x\rangle;$$

here $\langle j(af), x\rangle$ is the image in Q of the integer $\langle af, x\rangle$. So $\alpha\in A^0(n,m)$ and

$$\alpha(f_* x) = \langle af, x\rangle;$$

by Lemma 3·3, we have $a = \theta\alpha$. This completes the proof of Theorem 2·3.

Proof of Example 2·7. Let X, f, x_m be as in Example 2·7. For $k\neq 0$ the operation ψ^k may be interpreted as a map $\psi^k: K\to KQ$. By the proof of (2·3) we have

$$\langle \psi^k, f_* x_m\rangle = \langle \psi^k f, x_m\rangle.$$

By the ordinary calculation of ψ^k in CP^m we have

$$\begin{aligned}\psi^k(\xi-1) &= \xi^k - 1\\ &= ((\xi-1)+1)^k - 1\\ &= \sum_{1\leqslant j\leqslant m} \frac{k(k-1)\dots(k-j+1)}{1.2\dots j}(\xi-1)^j.\end{aligned}$$

So in this case we have

$$\langle \psi^k f, x_m\rangle = \frac{k(k-1)\dots(k-m+1)}{1.2\dots m}.$$

But by a similar calculation in S^{2r}, or directly, we have

$$\langle \psi^k, w^r\rangle = k^r.$$

So for a finite Laurent series

$$q(w) = \sum_r q_r w^r$$

with coefficients $q_r \in Q$, we have

$$\langle \psi^k, q(w)\rangle = \sum_r q_r k^r = q(k).$$

If we set $q(w) = f_* x_m$, we have

$$q(k) = \frac{k(k-1)\dots(k-m+1)}{1.2\dots m}.$$

But such a finite Laurent series is determined by its values for non-zero integral k. Therefore

$$q(w) = \frac{w(w-1)\dots(w-m+1)}{1.2\dots m}.$$

This proves Example 2·7.

Proof of Theorem 2·9. We rely on Lemma 3·1. It is well known – see for example (2) p. 24, Lemma 4·5 – that $K_0(\mathrm{MU})$ is generated as a ring by elements coming from $\mathrm{MU}(1)$. In order to have a map of spectra $\mathrm{MU}(1)\to\mathrm{MU}$, we must interpret $\mathrm{MU}(1)$ as the suspension-spectrum in which the space $\mathrm{MU}(1)$ occurs as the second term of the spectrum; that is, it is the spectrum $S^{-2}\wedge CP^\infty$. Then the generator x_{r+1} of filtration $r+1$ in $K_0(CP^\infty)$ gives a generator of filtration r in $K_0(\mathrm{MU}(1))$; in (2) this generator is called b_r^K. We have $b_0^K = 1$. By (2·7) the image of b_r^K in $K_0(K)$ is

$$p_r(w) = \frac{(w-1)(w-2)\dots(w-r)}{2.3\dots(r+1)} = w^{-1}\binom{w}{r+1}.$$

(The factor w^{-1} comes from the factor S^{-2}.) The proof in (2), by the Atiyah–Hirzebruch spectral sequence, actually shows that $K_0(\mathrm{MU})_{2m}$ is the Z-submodule generated by the products

$$b_{r_1}^K b_{r_2}^K \dots b_{r_\nu}^K$$

with $\qquad r_1 \geqslant 1,\quad r_2\geqslant 1,\quad \dots,\quad r_\nu \geqslant 1,\quad r_1+r_2+\dots+r_\nu \leqslant m.$

The map $t_*: K_0(\mathrm{MU})\to K_0(K)$ preserves products. By Lemma 3·1, $A_0(0,m)$ is the Z-submodule generated by the products

$$p_{r_1}(w)\,p_{r_2}(w)\dots p_{r_\nu}(w)$$

with $$r_1 \geqslant 1, \quad r_2 \geqslant 1, \quad \ldots, \quad r_\nu \geqslant 1, \quad r_1 + r_2 + \ldots + r_\nu \leqslant m.$$

This proves the result for $A_0(0, m)$; the general case follows by (2·6). This completes the proof of Theorem 2·9.

REFERENCES

(1) Adams, J. F. Lectures on generalised cohomology. In *Category Theory, Homology Theory and Their Applications*, III (Lecture Notes in Mathematics no. 99, Springer, 1969).

(2) Adams, J. F. *Stable Homotopy and Generalised Homology*, part II (Chicago University Press, 1974).

(3) Adams, J. F., Harris, A. S. and Switzer, R. M. Hopf algebras of cooperations for real and complex K-theory. *Proc. London Math. Soc.* 23 (1971), 385–408.

(4) Novikov, S. P. The methods of algebraic topology from the view-point of cobordism theories. *Izv. Akad. Nauk SSSR*, Serija Matematiceskaja, 31 (1967), 885–951.