University of Waterloo
Department of Pure Mathematics
Algebra Qualifying Examination
Groups and Rings
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REMINDER: The Qualifying Exams will be static open book, in the sense that consultation of all static pre-downloaded materials will be allowed. Examples of allowable aids include standard texts, and personal study notes. However, no other aids are permitted. In particular, discussing (over any medium) the exam with any person during the availability period, or using the internet or any other non-static tool to search questions or concepts, is not allowed.

Your submitted solutions must reflect your own understanding of the concepts being tested, in your own words.

When you submit a Qualifying Exam, you are agreeing to the following Academic Integrity Statement:

INTEGRITY STATEMENT

I declare the following statements to be true:

1. The work I submit here is entirely my own.
2. I have not used any unauthorized aids.
3. I have not discussed and will not discuss the contents of this examination with anyone until after the submission deadline.
4. I am aware that misconduct related to examinations can result in significant penalties, including failing the examination and suspension (this is covered in Policy 71: `https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71`)

Instructions: Attempt all four questions. Each quesion is worth 10 points so that the maximum mark is 40 points.

1. Let G and H be groups, and let $\phi\colon G \to H$ be a homomorphism of groups.

   (a) To each $g \in G$, associate a function $g\colon H \to H$ by $g(h) = \phi(g)h$. Determine if this is a left action of G on H. Justify your answer.

   (b) To each $g \in G$, associate a function $g\colon H \to H$ by $g(h) = h\phi(g)$. Determine if this is a left action of G on H. Justify your answer.

2. Let G be a finite group with $|G| = 52$. Let H be a subgroup of G with $|H| = 13$. Prove that H is a normal subgroup of G.

3. Let R be a ring. Let I be the right ideal of R that is generated by $\{xy - yx \mid x, y \in R\}$. Prove that I is a two-sided ideal of R.

4. For each of the three parts below, state whether the concluding statement is true or false. Justify your answers.

   (a) Let R be a ring with the multiplicative identity 1_R. Let $S \subset R$ be a subset of R that is a ring with exactly the same multiplication as R. If S has the multiplicative identity 1_S, then we must have $1_S = 1_R$.

   (b) Let R be a ring and let $p(x) \in R[x]$ be a polynomial of degree n with coefficients in R. Then $p(x)$ has at most n distinct roots in R.

   (c) Let $z_0 \neq 0$ be a complex root of a degree n polynomial with rational coefficients. Then $1/z_0$ is a root of a polynomial with rational coefficients and with degree less than or equal to n.