

University of Waterloo
Department of Pure Mathematics
Algebra Qualifying Examination
Linear Algebra
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REMINDER: The Qualifying Exams will be static open book, in the sense that consultation of all static pre-downloaded materials will be allowed. Examples of allowable aids include standard texts, and personal study notes. However, no other aids are permitted. In particular, discussing (over any medium) the exam with any person during the availability period, or using the internet or any other non-static tool to search questions or concepts, is not allowed.

Your submitted solutions must reflect your own understanding of the concepts being tested, in your own words.

When you submit a Qualifying Exam, you are agreeing to the following Academic Integrity Statement:

INTEGRITY STATEMENT

I declare the following statements to be true:

1. The work I submit here is entirely my own.
2. I have not used any unauthorized aids.
3. I have not discussed and will not discuss the contents of this examination with anyone until after the submission deadline.
4. I am aware that misconduct related to examinations can result in significant penalties, including failing the examination and suspension (this is covered in Policy 71: <https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71>)

Instructions: Attempt both questions. Each question is worth 10 points so that the maximum mark is 20 points.

1. Let n be a positive integer, and let d_n denote the determinant of the following $n \times n$ matrix:

$$M = \begin{bmatrix} 2 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 3 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 4 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 5 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & n+1 \end{bmatrix}.$$

What is $\lim_{n \rightarrow \infty} \frac{d_n}{n!}$?

2. Let k and n be positive integers. Let $G = \{A_1, A_2, \dots, A_k\}$ be a set of real $n \times n$ matrices such that G is a group under the usual matrix multiplication. Let $B = A_1 + A_2 + \cdots + A_k$, and let $\text{tr}(B)$ denote the trace of B .
- Prove that $A_i B = B$ for every $i \in \{1, 2, \dots, k\}$.
 - Prove that $B^2 = kB$.
 - Prove that if $\text{tr}(B) = 0$, then B is the zero matrix.