**University of Waterloo**
**Department of Pure Mathematics**

# Analysis Comprehensive Exam 2021
# Complex Analysis

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**September, 2021**

REMINDER: The Comprehensive Exams will be static open book, in the sense that consultation of all static pre-downloaded materials will be allowed. Examples of allowable aids include standard texts, and personal study notes. However, no other aids are permitted. In particular, discussing (over any medium) the exam with any person during the availability period, or using the internet or any other non-static tool to search questions or concepts, is not allowed.

Your submitted solutions must reflect your own understanding of the concepts being tested, in your own words.

When you submit a Comprehensive Exam, you are agreeing to the following Academic Integrity Statement:

INTEGRITY STATEMENT

I declare the following statements to be true:

1. The work I submit here is entirely my own.
2. I have not used any unauthorized aids.
3. I have not discussed and will not discuss the contents of this examination with anyone until after the submission deadline.
4. I am aware that misconduct related to examinations can result in significant penalties, including failing the examination and suspension (this is covered in Policy 71: `https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71`)

**Instructions:** Answer all questions.

1. Determine the number of zeros, with multiplicity, of the polynomial $f(z) = 1 + 4z^3 + z^{10} + 2z^{12}$ inside the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

2. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $t \in \mathbb{R}$, let $f_t$ denote the holomorphic function on $\mathbb{D}$ defined by

$$f_t(z) = \left(\frac{1+z}{1-z}\right)^{it}, \quad z \in \mathbb{D},$$

with respect to the principal branch of the logarithm.

(a) Show that the function $g(z) = \dfrac{1+z}{1-z}$ maps $\mathbb{D}$ onto $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

(b) Show there is $C > 0$ such that for all $t \in \mathbb{R}$,

$$\sup_{z \in \mathbb{D}} |f_t(z)| < C^t,$$

(c) Deduce that for every infinite bounded subset $X \subseteq \mathbb{R}$ there is a sequence $(t_n)_{n=1}^{\infty}$ of distinct points in $X$ such that the sequence $(f_{t_n})_{n=1}^{\infty}$ converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function on $\mathbb{D}$.

3. Evaluate

$$\int_0^{\pi} \frac{\cos\theta}{3 + \cos\theta} \, d\theta.$$