University of Waterloo
Department of Pure Mathematics

Analysis Comprehensive Exam 2021
Topology and Real Analysis

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REMINDER: The Comprehensive Exams will be static open book, in the sense that consultation of all static pre-downloaded materials will be allowed. Examples of allowable aids include standard texts, and personal study notes. However, no other aids are permitted. In particular, discussing (over any medium) the exam with any person during the availability period, or using the internet or any other non-static tool to search questions or concepts, is not allowed.

Your submitted solutions must reflect your own understanding of the concepts being tested, in your own words.

When you submit a Comprehensive Exam, you are agreeing to the following Academic Integrity Statement:

INTEGRITY STATEMENT

I declare the following statements to be true:

1. The work I submit here is entirely my own.
2. I have not used any unauthorized aids.
3. I have not discussed and will not discuss the contents of this examination with anyone until after the submission deadline.
4. I am aware that misconduct related to examinations can result in significant penalties, including failing the examination and suspension (this is covered in Policy 71: `https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71`)

Instructions: Answer all questions.

1. Let A be a $d \times d$ positive semi-definite matrix over $\mathbb{C}$. Show that

$$\lim_{n\to\infty} (\mathrm{Tr}(A^n))^{1/n} = \max\{\lambda : \lambda \text{ is an eigenvalue of } A\},$$

where Tr denotes the trace.

2. Let $M_d(\mathbb{C})$ denote the space of complex $d \times d$ matrices, equipped with the trace norm $\|A\|_2 = (\mathrm{Tr}(A^*A))^{1/2}$, where A^* denotes the adjoint of A with entries $(A^*)_{ij} = \overline{A_{ji}}$.

   (a) Let $U \in M_d(\mathbb{C})$ be unitary. Show that $\|UAU^*\|_2 = \|A\|_2$ for $A \in M_d(\mathbb{C})$.

   (b) Let $M_d(\mathbb{C})_{sa} = \{A \in M_d(\mathbb{C}) : A^* = A\}$ denote the set of self-adjoint matrices in $M_d(\mathbb{C})$, equipped with the metric $d(A, B) = \|A - B\|_2$. Show that the subset of $M_d(\mathbb{C})_{sa}$ consisting of self-adjoint matrices with rational eigenvalues is dense in $M_d(\mathbb{C})_{sa}$.

3. For a compact metric space X, let $C(X)$ denote the space of real-valued continuous functions on X, equipped with the uniform norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

   Let $I = [0, 1]$. For $k \in C(I \times I)$ and f in $C(I)$, define $T_k f$ by

$$T_k f(x) = \int_0^1 k(x, y) f(y)\, dy, \qquad \text{for } x \in I.$$

   (a) For $k \in C(I \times I)$, show that $T_k f$ is continuous for all $f \in C(I)$.

   (b) Deduce that T_k is a linear map on $C(I)$ satisfying $\|T_k\|_{op} \le \|k\|_\infty$, where

$$\|T_k\|_{op} = \sup_{\substack{f\in C(I)\\ \|f\|_\infty \le 1}} \|T_k f\|_\infty$$

   is the operator norm.

   (c) A function $k \in C(I \times I)$ is *elementary* if there exist $f_1, f_2 \in C(I)$ such that $k(x, y) = f_1(x) f_2(y)$ for all $(x, y) \in I \times I$. Show that if k is a linear combination of elementary functions, then T_k has finite rank.

   (d) Show that the elementary functions span a dense subset of $C(I \times I)$. Deduce that for every $k \in C(I \times I)$, the operator T_k is an operator norm limit of finite rank operators.