Pure Mathematics Complex Analysis Qualifying Examination
University of Waterloo
September 14, 2023

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. *(a)* State Rouché's Theorem.

   (b) Consider the polynomial P defined by putting

$$P(z) := \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^{100}}{100!}, \quad z \in \mathbb{C}.$$

   By using Rouché's Theorem, prove the following statement:

> For every $\lambda \in \mathbb{C}$ such that $|\lambda| > 2$,
> the equation $P(z) = \lambda z - 1$ has exactly one solution
> in the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. *(a)* State Liouville's Theorem.

   (b) Let f be an entire function satisfying $\mathrm{Im} f(z) \leq 2023$. Prove that f is constant.

   (c) Prove that every non-constant polynomial p with complex coefficients has at least one root in $\mathbb{C}$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. *(a)* Let f be an analytic function on the punctured disk $0 < |z - z_0| < r$. Define the *residue of f at z_0*.

(b) State the Cauchy Residue Theorem.

(c) Let $\lambda > 0$ and let $f_\lambda(z) = \frac{e^{iz}}{z^2+\lambda^2}$. Compute the residues of $f_\lambda(z)$ at each of its poles.

(d) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + \lambda^2} dx$$

using the Cauchy Residue Theorem. Fully justify each of your steps.

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