Pure Mathematics Linear Algebra Qualifying Examination
University of Waterloo
September 21, 2023

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let A be a complex 4×4 matrix with minimal polynomial $(x-1)(x-3)^2$. Find all possible Jordan canonical forms for the matrix A (to avoid repetitions, you can assume that the diagonal entries of the Jordan canonical form are increasing in magnitude). Justify your answer.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. A matrix B is a square root of a matrix A if $B^2 = A$.

   (a) Find a 2×2 matrix with an infinite number of square roots.

   (b) Let A be a real symmetric matrix that is positive definite. Show that A has a unique square root B that is symmetric and positive definite. Hint: For uniqueness, show that B preserves the eigenspaces of A.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let V be a finite dimensional vector space of dimension n. Recall that if W_1, W_2 are two subspaces of V,

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Let $W_1, \ldots, W_k$ be $k < n$ subspaces of V of dimension $n - 1$. Show that

$$\dim(W_1 \cap \cdots \cap W_k) \geq n - k.$$

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