Pure Mathematics Measure Theory and Fourier Analysis Qualifying Examination
University of Waterloo
September 19, 2023

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let λ be the Lebesgue measure on the interval $[0,1]$. For $p \in [1,\infty)$, we denote by $(L^p(\lambda), \|\cdot\|_p)$ the associated Banach space of p-integrable functions on $[0,1]$.

   (a) Let $p, q \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. State *Hölder's inequality* concerning functions from the spaces $L^p(\lambda)$ and $L^q(\lambda)$.

   (b) Prove that for $p_1 \leq p_2$ in $[1,\infty)$, one has the inclusion $L^{p_2}(\lambda) \subseteq L^{p_1}(\lambda)$, and the inequality of norms
$$||f||_{p_1} \leq ||f||_{p_2}, \quad \forall\, f \in L^{p_2}(\lambda).$$

   (c) Fix $T > 1$ and a function $f \in L^T(\lambda)$ and let $\varphi : [1,T] \to \mathbb{R}$ be defined by
$$\varphi(p) := ||f||_p, \quad \forall\, p \in [1,T].$$
   Prove that φ is a continuous function. (Hint: Apply the Lebesgue Dominated Convergence Theorem).

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. In this problem we use the following definition:

Definition. Let (X, d) be a metric space, let $\mathfrak{B}$ be the Borel sigma-algebra of (X, d), and let $\mu : \mathfrak{B} \to [0, 1]$ be a Borel probability measure. We say that μ is *closed regular* if the following holds:

$$\text{(Cl-Reg)} \quad \begin{cases} \text{for every } B \in \mathfrak{B} \text{ one has} \\ \mu(B) = \inf\{\mu(G) \mid G \text{ open}, G \supseteq B\} \; = \; \sup\{\mu(F) \mid F \text{ closed}, F \subseteq B\}. \end{cases}$$

The aim of this problem is to prove that *every* Borel probability measure μ on (X, d) is automatically closed regular. To that end, let $\mathfrak{A} \subseteq \mathfrak{B}$ be the collection of subsets of X given by

$$\mathfrak{A} = \left\{ A \in \mathfrak{B} \;\middle|\; \begin{array}{l} \mu(A) = \inf\{\mu(G) \mid G \text{ open}, G \supseteq A\} \text{ and} \\ \mu(A) = \; \sup\{\mu(F) \mid F \text{ closed}, F \subseteq A\} \end{array} \right\}.$$

(a) Prove that $\mathfrak{A}$ contains every closed subset of X.

(b) Prove that $\mathfrak{A}$ is closed under complements.

(c) You may assume, without proof, that $\mathfrak{A}$ is closed under countable unions. Conclude that $\mathfrak{A} = \mathfrak{B}$, and that μ is closed regular.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let $\mathcal{X}$ be a Hilbert space over $\mathbb{R}$ and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis for $\mathcal{X}$.

   (a) Let $\eta \in \mathcal{X}$. Define the Fourier coefficients of η with respect to the orthonormal basis $(\xi_n)_{n\geq 0}$, and state the Parseval identity concerning these coefficients.

For the remainder of this problem, we take $\mathcal{X} = L^2([-\pi,\pi])$, and we accept the fact that this Hilbert space has an *orthonormal system* $(\varphi_n)_{n\geq 0}$ defined as follows: φ_0 is the function identically equal to $\frac{1}{\sqrt{2\pi}}$, and then for every $k \geq 1$ we define φ_{2k-1} and φ_{2k} by putting

$$\varphi_{2k-1}(t) = \frac{1}{\sqrt{\pi}}\sin(kt) \;\text{ and }\; \varphi_{2k}(t) = \frac{1}{\sqrt{\pi}}\cos(kt), \quad \forall\, t \in [-\pi,\pi].$$

We also consider the function $\eta \in L^2([-\pi,\pi])$ defined by putting $\eta(t) = t$ for all $t \in [-\pi,\pi]$.

   (b) Explain why the orthonormal system $(\varphi_n)_{n\geq 0}$ is in fact an *orthonormal basis* of $\mathcal{X}$. (Hint: Use the Stone Weierstrass Theorem)

   (c) Let $(c_n)_{n\geq 0}$ be the Fourier coefficients of η with respect to $(\varphi_n)_{n\geq 0}$. Prove that $c_n = 0$ for every even $n \geq 0$ and compute the explicit value of c_n for an odd $n \geq 1$.

   (d) By using your answers from above, determine, with justification, the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

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