Pure Mathematics Topology and Real Analysis Qualifying Examination
University of Waterloo
September 12, 2023

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let (X,d) be a metric space.

   (a) Define what it means for (X,d) to be totally bounded.

   (b) Prove that every totally bounded metric space is separable.

   (c) Let (X,d) be a metric space and $\epsilon > 0$. Suppose that there exists an uncountable family $(x_i)_{i\in I}$ such that $d(x_i,x_j) \geq \epsilon$ for all $i \neq j \in I$. Prove that (X,d) is not separable.

   (d) Let (X,d) be the metric space of bounded sequences of real numbers $(\alpha_i)_{i\in\mathbb{N}}$ equipped with the metric
   $$d\big((\alpha_i)_{i\in\mathbb{N}},(\beta_i)_{i\in\mathbb{N}}\big) = \sup_{i\in\mathbb{N}} |\alpha_i - \beta_i|$$
   for $(\alpha_i)_{i\in\mathbb{N}},(\beta_i)_{i\in\mathbb{N}} \in X$. Is X separable? Justify your answer.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. *(a)* State the Baire category theorem.

(b) Let $(X, ||\cdot||)$ be a Banach space. Suppose that C is a closed subset of X which has the following property:

$$\text{For every } x \in X, \text{ there exists } n \in \mathbb{N} \text{ such that } \tfrac{1}{n}x \in C.$$

Prove that C has non-empty interior.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let $(X, \mathcal{T})$ be a topological space.

   (a) State the definition of a *basis of open sets* for $(X, \mathcal{T})$.

   (b) State the definition of what it means for $(X, \mathcal{T})$ to be *second countable.*

   For parts (c) and (d), we consider the notion of *condensation point.* Let $(X, \mathcal{T})$ be a topological space, let A be a subset of X, and let x be a point of X. We say that x is a *condensation point* for A to mean that

$$[G \in \mathcal{T},\, G \ni x] \;\Rightarrow\; [G \cap A \text{ is an infinite uncountable set}].$$

   (c) Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let C denote the set of all condensation points of A. Prove that C is a closed (possibly empty) subset of X.

   (d) Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let C denote the set of all condensation points of A. Suppose you know that $(X, \mathcal{T})$ is second countable, and that A is an infinite uncountable set. Prove that $C \neq \emptyset$.

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