Pure Mathematics Linear Algebra Qualifying Examination
University of Waterloo
September 19, 2024

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let $R = \mathbb{C}[t]$, and let V be the complex vector space with basis $1, t, t^2 \in R$. Consider the linear transformation $\phi : V \to V$ given by

$$\phi(a + bt + ct^2) = 2a + (b + 3c)t + (3b + c)t^2.$$

Is ϕ diagonalizable? If so, find a basis in which it is diagonal.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. Let V be a finite dimensional vector space, and $T : V \to V$ a linear transformation. Show the following statements.

   (a) If T is invertible, then the minimal polynomial of T has non-zero constant term.

   (b) If the minimal polynomial of T has non-zero constant term, then T is invertible.

   (c) If T is invertible, then T^{-1} can be expressed as a polynomial in T.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let A be an $n \times n$ real matrix.

   (a) Show that if all the eigenvalues of A are zero, then A is nilpotent.

   (b) Show that if A is nilpotent, then all the eigenvalues of A are zero.

   (c) Show that if A is nilpotent, then $\det(A + I_n) = 1$, where I_n is the $n \times n$ identity matrix.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

4. Let V be a finite dimensional real vector space with a positive definite inner product $(\cdot, \cdot)$. Suppose $\{v_1, \ldots, v_k\}$ is a set of k vectors in V satisfying

$$(v_i, v_j) < 0 \text{ for all } i \neq j.$$

Show that the linear span of $v_1, \ldots, v_k$ has dimension at least $k - 1$.

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