Pure Mathematics Measure Theory and Fourier Analysis Qualifying Examination
University of Waterloo
September 17, 2024

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. In this question, $(X, \mathfrak{A})$ is a measurable space. We consider the space of Borel functions

$$\mathrm{Bor}(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is } \mathfrak{A}/\mathfrak{B}_{\mathbb{R}}\text{-measurable}\},$$

where $\mathfrak{B}_{\mathbb{R}}$ denotes the Borel sigma-algebra of $\mathbb{R}$.

(a) Let $f : X \to \mathbb{R}$ be a function. Suppose we found a collection of Borel sets $\mathfrak{C} \subseteq \mathfrak{B}_{\mathbb{R}}$ such that the sigma-algebra generated by $\mathfrak{C}$ is $\mathfrak{B}_{\mathbb{R}}$, and such that $f^{-1}(C) \in \mathfrak{A}$ for every $C \in \mathfrak{C}$. Prove that $f \in \mathrm{Bor}(X, \mathbb{R})$.

(b) Let $f : X \to \mathbb{R}$ be a function for which the following is known:

$f^{-1}(J) \in \mathfrak{A}$ whenever $J \subseteq \mathbb{R}$ is a compact interval of length 1 (that is, $J = [a, a+1]$ for some $a \in \mathbb{R}$).

Prove that $f \in \mathrm{Bor}(X, \mathbb{R})$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. In this question, (X,d) is a metric space, and we let $\mathfrak{B} \subseteq 2^X$ be the Borel sigma-algebra of (X,d). We consider the set of probability measures $\text{Prob}(X,d) := \{\mu : \mathfrak{B} \to [0,1] \mid \mu \text{ is a probability measure}\}$.

(a) Let $\mu \in \text{Prob}(X,d)$ and let F be a closed subset of X. Prove: for every $\varepsilon > 0$, there exists an open set $G \subseteq X$ such that $G \supseteq F$ and $\mu(G) < \mu(F) + \varepsilon$.

(b) Let μ and $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ be in $\text{Prob}(X,d)$ and suppose that the following holds:

For every bounded continuous function $f : X \to \mathbb{R}$
one has that $\lim_{n\to\infty} \int_X f \, d\mu_n = \int_X f \, d\mu$.

Let F be a closed subset of X. Prove that $\text{limsup}_{n\to\infty} \mu_n(F) \leq \mu(F)$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. In this question we let $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ be the standard orthonormal basis of the space $L^2\big([-\pi,\pi]\big)$. That is, for every $k \in \mathbb{N}$ we put

$$\varphi_{2k-1}(t) = \frac{1}{\sqrt{\pi}}\sin(kt) \quad \text{and} \quad \varphi_{2k}(t) = \frac{1}{\sqrt{\pi}}\cos(kt), \quad \forall\, t \in [-\pi,\pi],$$

and we also let $\varphi_0 : [-\pi,\pi] \to \mathbb{R}$ be constantly equal to $1/\sqrt{2\pi}$.

We fix throughout the question a function $f \in L^1\big([-\pi,\pi]\big)$, which is known to have $\int_{-\pi}^{\pi} f(t)\,dt = 1$.

(a) Define the Fourier coefficients $(c_n)_{n\in\mathbb{N}\cup\{0\}}$ of the function f.

(b) State the Riemann-Lebesgue theorem.

(c) Prove that the limits:

$$\alpha \;:=\; \lim_{k\to\infty} \int_{-\pi}^{\pi} f(t)\sin^2(kt)\;dt \quad \text{and} \quad \beta \;:=\; \lim_{k\to\infty} \int_{-\pi}^{\pi} f(t)\cos^2(kt)\;dt$$

exist, and determine their values.

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