Pure Mathematics Topology and Real Analysis Qualifying Examination
University of Waterloo
September 10, 2024

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Find $\lim_{n\to\infty}(\int_0^{\frac{\pi}{2}} \sin^n(x)dx)^{1/n}$. (Prove any general statements you use.)

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. (a) State a form of the Baire category theorem.

(b) Let $f_1, f_2, \ldots, f_n, \ldots$ be continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and suppose that the following holds:

$$\forall\, x \in \mathbb{R},\ \exists\, n \geq 1 \text{ such that } f_n(x) \in \mathbb{Q}.$$

Prove: there exist $a < b$ in $\mathbb{R}$ and $n \geq 1$ such that f_n is constant on the interval (a, b).

(c) Show by example that the conclusion of part (b) no longer holds if we do not assume the functions f_n to be continuous.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let X be a compact Hausdorff topological space. Suppose that $\mathcal{U}$ is a countable basis of open sets for X, where every $U \in \mathcal{U}$ is at the same time open and closed. Suppose, moreover, that $X \in \mathcal{U}$ and that $[U_1, U_2 \in \mathcal{U}] \Rightarrow [U_1 \cap U_2 \in \mathcal{U}]$.

(a) Prove: for every $U \in \mathcal{U}$, the indicator function $1_U : X \to \mathbb{R}$ belongs to the space $C(X, \mathbb{R})$ of continuous functions from X to $\mathbb{R}$.

(b) Let $\mathcal{A} = \mathrm{span}\{1_U \mid U \in \mathcal{U}\}$. Prove that $\mathcal{A}$ is a unital subalgebra of $C(X, \mathbb{R})$.

(c) Let $\mathcal{A}$ be as above. Prove: $\mathcal{A}$ is dense in $C(X, \mathbb{R})$, in the topology of uniform convergence.

(d) Find a countable set $\mathcal{Q} \subseteq C(X, \mathbb{R})$ which is dense in $C(X, \mathbb{R})$, in the topology of uniform convergence.

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