

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. The two parts of this problem are *not* related.

(a) Consider the region

$$A = \{z = x + iy \mid |z| < 1, y \geq 0\}$$

in the complex plane. Can you find a complex-valued function f defined on A whose restriction to $y = 0$ is $f(x) = |x|$ and which is holomorphic in the interior of A ? Justify your answer.

(b) Can you find a sequence of holomorphic functions on the unit disk $\{z \mid |z| < 1\}$ that converges to the function $g(z) = \frac{1}{z}$ locally uniformly away from 0? Justify your answer.

2. Given a polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ where $n \geq 1$, show that

$$\max_{|z| \leq 1} |P(z)| \geq 1.$$

3. Let U be open in $\mathbb{C}$.

- (a) Suppose $g: U \rightarrow \mathbb{C}$ is *continuous* on U . Let $z_0 \in U$ such that $g(z_0) \neq 0$. Prove that there exists an open disc $B_\epsilon(z_0)$ in U centred at z_0 such that $\int_\Gamma g(z)dz \neq 0$ whenever Γ is a straight line segment whose two end points lie in $B_\epsilon(z_0)$.
- (b) Suppose $f: U \rightarrow \mathbb{C}$ is *holomorphic* on U , and $f'(z_0) \neq 0$ for some $z_0 \in U$. Prove that there exists an open disc $B_\epsilon(z_0)$ in U centred at z_0 on which f is one-to-one.

Note: You must use part [a] here. You are not permitted to use the inverse function theorem.

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4. Let $w = f(z)$ be an entire nonconstant function. Prove that its image $f(\mathbb{C})$ intersects *every nonempty open set* in the w -plane.

Hint: Do this by contradiction. Find a way to use Liouville's Theorem.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

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