Pure Mathematics Measure Theory Qualifying Examination
University of Waterloo
September 16, 2025

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, and consider the function

$$f(x) = \frac{\pi}{\sin(\pi\alpha)} e^{i(\pi - x)\alpha}$$

on $[0, 2\pi]$.

(a) Show that the Fourier series of f is given by

$$f \sim \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}.$$

(b) Evaluate $\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2}$, and conclude that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^2} = \pi^2.$$

2. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with measure $\mu(E) < +\infty$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real-valued functions defined on E, which converges to f almost everywhere on E.

   (a) Given $\epsilon > 0$ and $\delta > 0$, show that there exists a subset $A \subseteq E$ with $\mu(A) < \delta$ and an $N_0 \in \mathbb{N}$ such that for all $x \notin A$ and all $n \geq N_0$, we have $|f_n(x) - f(x)| < \epsilon$.
   Hint: Consider $G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$.

   (b) Given any $\eta > 0$, prove that there exists a subset $B \subseteq E$ with $\mu(B) < \eta$, such that $(f_n)_{n\in\mathbb{N}}$ converges *uniformly* to f on $E \setminus B$.

3. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with measure $\mu(E) < +\infty$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real-valued functions defined on E, which converges to f almost everywhere on E. Suppose there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all $n \in \mathbb{N}$.

   (a) Explain why f is Lebesgue integrable on E.

   (b) Prove that
   $$\lim_{n\to\infty} \int_E f_n = \int_E f$$
   without directly applying the Dominated Convergence Theorem. Instead, use part (b) of the previous problem.

4. Given a sequence of measurable functions f_n over $[0,1]$ with $|f_n(x)| < \infty$ for a.e. $x \in [0,1]$, show that there exists a sequence of numbers c_n so that

$$\frac{f_n(x)}{c_n} \to 0$$

for a.e. $x \in [0,1]$ as $n \to \infty$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

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