Pure Mathematics Real Analysis Qualifying Examination
University of Waterloo
September 9, 2025

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Can you approximate the function $|\sin(x)|$ over $[0, 1]$ uniformly with

   (a) a sequence of polynomials with uniformly bounded degrees?

   (b) a sequence of nonnegative polynomials?

   Justify your answers.

2. Does the sequence of functions $\{\sin(nx)\}_{n=1}^{\infty}$ have a uniformly convergent subsequence over $[0, 1]$? Justify your answer.

3. Let (X, d) be a metric space, and let $E \subseteq X$ be a nonempty *bounded* subset. We define the *diameter* of E, denoted $\operatorname{diam} E$, to be

$$\operatorname{diam} E = \sup\{d(x, y) : x, y \in E\}.$$

(a) Prove that this supremum exists, and that $\operatorname{diam} E \geq 0$.

(b) Prove that its closure $\overline{E}$ is also bounded, and that $\operatorname{diam} E \leq \operatorname{diam} \overline{E}$.

(c) Prove that $\operatorname{diam} E = \operatorname{diam} \overline{E}$.

4. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuous on $\mathbb{R}^n$. Suppose that f has the following property: for any $\epsilon > 0$, there exists $M > 0$ such that if $\|\mathbf{x}\| > M$, then $|f(\mathbf{x})| < \epsilon$. (We say that f *goes to zero at infinity.*)

   (a) Show that f is bounded on $\mathbb{R}^n$.

   (b) Suppose there exists some point $\mathbf{x}_0 \in \mathbb{R}^n$ such that $f(\mathbf{x}_0) > 0$. Prove that there exists a point $\tilde{\mathbf{x}}$ such that $f(\mathbf{x}) \leq f(\tilde{\mathbf{x}})$ for all $\mathbf{x} \in \mathbb{R}^n$. (That is, show that f *attains a global maximum* on $\mathbb{R}^n$.)

   [Note that we cannot *directly* use the Extreme Value Theorem because $\mathbb{R}^n$ is not compact. Find a creative way to use it nevertheless.]

   (c) Prove that f is uniformly continuous on $\mathbb{R}^n$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

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