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A Bernstein-Type Formula for Projective Representations of S_n and A_n

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The formula in the theorem below was motivated by another due to Bernstein for the ordinary irreducible representations of S_n [Z, p. 69]. Taken as a definition, it yields a quick derivation of the irreducible projective representations as linear combinations (in the representation ring) of certain well-known representations induced from Clifford modules. Schur's original derivation [Sc] involved some difficult symmetric function identities. Recently Stembridge [St] has given another derivation, which depends on substantial combinatorial results of Sagan [Sa] and Worley [W] concerning shifted Young tableaux.

Bernstein's formula was presented by Zelevinski in the abstract setting of PSH-algebras. The following is a list of basic results which we need. This makes it clear that the theorem is valid in the context of L-PSH-algebras, though we shall not prove it in that setting. Details concerning L-PSH-algebras and proofs of the elementary facts below may be found in [BH], HH1, HH2].

SUMMARY OF RESULTS NEEDED

I. The group $\tilde{S}_n$, with presentation

$$\langle z, t_1, t_2, ..., t_{n-1}; z^2=1, t_i^2=z=(t_it_{i+1})^3, t_jt_i=zt_it_j \text{ for } j>i+1\rangle,$$

is a double cover of S_n via the map π sending t_i to the transposition $(i\ i+1)$.

II. The difference between the numbers of conjugacy classes in $\tilde{S}_n$ and in S_n is $2\#\mathscr{D}'_n + \#\mathscr{D}''_n$, where $\mathscr{D}'_n$ (resp. $\mathscr{D}''_n$) is the set of partitions of n into distinct parts among which the number of even parts is odd (resp. even). If $\tilde{A}_n := \pi^{-1}A_n$, the corresponding difference for $\tilde{A}_n$ and A_n is $2\#\mathscr{D}''_n + \#\mathscr{D}'_n$.

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III. For $n \geq 4$, the projective representations of S_n are in natural 1–1 correspondence with those representations of $\tilde{S}_n$ in which z acts as -1.

IV. Disregarding a few exceptional cocycles which exist when $n=6$ and 7, for $n \geq 4$ the projective representations of A_n (up to equivalence) are in 1–1 correspondence with the isomorphism classes of objects (V_0, V_1, ψ), where V_0 and V_1 are complex vector spaces, and ψ is a representation of $\tilde{S}_n$ on $V_0 \oplus V_1$ in which z acts as -1 and in which $s \in \tilde{S}_n$ maps V_i to either V_i or V_{i+1} according to whether $s \in \tilde{A}_n$ or $s \notin \tilde{A}_n$.

(In V and VI, $\tilde{S}_n$ may be replaced by any object (G, z, σ) as in [HH1, Sect. 1].

V. Define, for all $n \geq 0$, the $\mathbb{Z}/2$-graded group $T^*(\tilde{S}_n) = T^0(\tilde{S}_n) \oplus T^1(\tilde{S}_n)$ as follows. The Grothendieck group generated by isomorphism classes of "negative" representations of $\tilde{S}_n$ as in III is denoted $T^1(\tilde{S}_n)$. That generated by "negative $\mathbb{Z}/2$-graded" representations of $\tilde{S}_n$ as in IV is denoted $T^0(\tilde{S}_n)$. Then $T^*(\tilde{S}_n)$ is a $\mathbb{Z}/2$-graded module over the $\mathbb{Z}/2$-graded ring $L = \mathbb{Z}[\lambda]/(\lambda^3 - 2\lambda)$. Note that L^0 has canonical ("positive") basis $\{1, \rho\}$ over $\mathbb{Z}$, where $\lambda^2 = 1 + \rho$, and L^1 has canonical basis $\{\lambda\}$. The action of λ corresponds to inducing and restricting between $\tilde{S}_n$ and $\tilde{A}_n$ (except for small n).

VI. Over L, $T^*\tilde{S}_n$ is free. An L-basis (the "special irreducibles") is concocted by choosing, for each irreducible x for which $\rho x \neq x$, exactly one of x or ρx. Thus $T^*\tilde{S}_n$ has rank over L equal to $\#\mathscr{D}'_n + \#\mathscr{D}''_n$, by II.

Taking such a basis to be orthonormal, we obtain a symmetric L-bilinear inner product $\langle\ ,\ \rangle$: $T^*(\tilde{S}_n) \times T^*(\tilde{S}_n) \to L$. This is independent of the choices above. The only other orthonormal bases besides these are obtained by multiplying some or all of the special irreducibles by -1.

VII. Let φ: $\tilde{S}_i \curlyvee \tilde{S}_j \to \tilde{S}_{i+j}$ be the inverse image under π of the usual embedding $S_i \times S_j \to S_{i+j}$ (of a Young subgroup). Let φ_* and φ^* denote respectively inducing and restricting. Then the product and coproduct maps defined below make $H := \oplus_{n \geq 0} T^*(\tilde{S}_n)$ into a self-dual associative graded Hopf algebra over L:

$$\mu: T(\tilde{S}_i) \otimes_L T^*(\tilde{S}_j) \xrightarrow{\cong} T^*(\tilde{S}_i \curlyvee \tilde{S}_j) \xrightarrow{\varphi_*} T^*(\tilde{S}_{i+j})$$

$$\Delta: T^*(\tilde{S}_{i+j}) \xrightarrow{\varphi^*} T^*(\tilde{S}_i \curlyvee \tilde{S}_j) \cong T^*(\tilde{S}_i) \otimes_L T^*(\tilde{S}_j).$$

For the isomorphism above see [HH1, 2.24]. One has a pseudo-commutative law

$$xy = \rho^{\varepsilon\delta + ij} yx \qquad \text{for} \quad x \in T^\varepsilon(\tilde{S}_i),\ y \in T^\delta(\tilde{S}_j). \tag{COM}$$

Its dual holds for the coproduct. For $x \in H$, define $x^*: H \to H$ by

$$\langle x^*(y), z \rangle = \langle y, xz \rangle.$$

Reciprocity gives the formula

$$\langle ab, c \rangle = \langle\!\langle a \otimes b, \Delta c \rangle\!\rangle,$$

where $\langle\!\langle\ ,\ \rangle\!\rangle$ is the inner product on $H \otimes_L H$ determined by

$$\langle\!\langle a \otimes b, a' \otimes b' \rangle\!\rangle = \langle a, a' \rangle \cdot \langle b, b' \rangle.$$

VIII. There exist special irreducibles $h_n \in T^{n+1}(\tilde{S}_n)$ defined uniquely, up to replacement by ρh_n ("ρ-uniquely"), by the equation

$$h_1 h_{n-1} = \lambda h_n + u_n,$$

where u_n is a special irreducible for $n \geq 3$, and $u_2 = 0$. These satisfy the squaring relation

$$h_n^2 = (-1)^{n+1} \lambda \left[h_{2n} + \lambda \sum_{i=1}^{n-1} (-1)^i h_{2n-i} h_i \right], \tag{SQ}$$

and have coproduct

$$\Delta h_n = h_n \otimes 1 + 1 \otimes h_n + \lambda \sum_{i=1}^{n-1} h_i \otimes h_{n-i}.$$

It follows that

$$h_n^*(xy) = h_n^*(x) y + \rho^{n(i+\varepsilon)+\varepsilon} x h_n^*(y) + \lambda \sum_{i=1}^{n-1} h_i^*(x)\, h_{n-i}^*(y)$$

for x in $T^\varepsilon(\tilde{S}_i)$.

Notes. The h_n can be defined explicitly as the classes of irreducible Clifford modules [HH1, St] or equivalently given as lengthy Kronecker products of 2×2 matrices [Sc]. But their existence and above properties also follow formally in the L-PSH-algebra setting [BH, 4.3]. The above facts are all quite elementary. In [HH1, HH2] Hopf algebra theory is then used to prove that, as an algebra, H is generated by $\{h_1, h_2, \ldots\}$, subject to only the relations (COM) and (SQ). This is longer and more tortuous than the argument below and does not yield explicit formulae for the irreducibles in terms of products fo the h_i. The formula and arguments below yield a shorter non-Hopfian proof of the above algebra structure.

Definition. For each $n \geq 1$, define

$$R_n: H \to H$$

by

$$R_n(x) = h_n x + \lambda \sum_{i>0} (-1)^i (h_{n+i})(h_i^* x).$$

It is clear that for each x the summation is finite. The effect of R_n on $\mathbb{N}$-degree is to raise it by n. It changes $\mathbb{Z}/2$-degree by $n+1$.

DEFINITION. For each sequence $\alpha=(n_1, n_2, ..., n_s)$ of positive integers, define the element $l_\alpha := R_{n_1} R_{n_2} \cdots R_{n_s}(1) \in T^{n_1+\cdots+n_s+s}(\tilde{S}_{n_1+\cdots+n_s})$.

Note. We shall need only the case $n_1 > n_2 > \cdots$. In fact, for other α, the l_α are determined, since $R_a R_b = -\rho^{a+b} R_b R_a$. For example, $l_\alpha = 0$ if (and only if) the n_i are not all distinct.

THEOREM. (i) *The elements* l_α, ρl_α, *and* λl_α, *as* α *ranges over strictly decreasing sequences, give all the irreducible elements of* H (*without repeats*).

(ii) *Ordering the set of all decreasing* α *by reverse lexicographic order, we have*

$$\begin{pmatrix} \vdots \\ h_\alpha \\ \vdots \end{pmatrix} = N \begin{pmatrix} \vdots \\ l_\alpha \\ \vdots \end{pmatrix},$$

where N *has the form*

$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ * & & & 1 \end{pmatrix}.$$

Here, for $\alpha=(n_1, n_2, ...)$, h_α *means the product* $h_{n_1} h_{n_2} \cdots$.

Schur's work essentially contains (ii).

Proof. It suffices to prove (A) and (B) below:

(A) There exist $\mu_{\alpha\beta} \in L$ with $l_\alpha = h_\alpha + \lambda \sum_{\beta<\alpha} \mu_{\alpha\beta} h_\beta$;

(B) $\langle l_\gamma, h_\alpha \rangle = \begin{cases} 1 & \text{if } \alpha=\gamma \\ 0 & \text{if } \alpha<\gamma. \end{cases}$

For, given (A) and (B), if $\alpha \leqslant \gamma$, we have

$$\langle l_\gamma, l_\alpha \rangle = \langle l_\gamma, h_\alpha \rangle + \lambda \sum_{\beta<\alpha} \mu_{\alpha\beta} \langle l_\gamma, h_\beta \rangle = \langle l_\gamma, h_\alpha \rangle,$$

which is 0 if $\alpha<\gamma$ and 1 if $\alpha=\gamma$. Thus, using II, $\{l_\alpha\}$ is the (ρ-unique) L-basis of special irreducibles, yielding (i). Inverting (A) gives (ii).

Aside. Note that the entries below the diagonal in N are divisible by λ in L. A combinatorial formula for them as sums of powers of λ is given at the end of this paper.

By induction on the length of α, the proof of (A) is immediate from the definitions of l_α and R_n.

To prove (B), since $\langle l_\gamma, h_{n_1} \cdots h_{n_s} \rangle = h_{n_s}^* \cdots h_{n_1}^*(l_\gamma)$, it suffices to prove (C) and (D) below, where $a > b > \cdots$:

(C) $h_a^*(l_{a,b,...}) = l_{b,...}$;

(D) $h_{a+n}^*(l_{a,b,...}) = 0$ for $n>0$.

We shall use the identity $\lambda x^* y^* = \lambda y^* x^*$ several times, and assume inductively that (C) and (D) hold for sequences shorter than $(a, b, ...)$.

Proof of (C).

$$\begin{aligned} & h_a^*(l_{a,b,...}) \\ &= h_a^*(h_a l_{b,...}) + \lambda \sum_{i>0} (-1)^i h_a^*[(h_{a+i})(h_i^* l_{b,...})] \\ &= h_a^*(h_a)\, l_{b,...} + \rho h_a h_a^*(l_{b,...}) + \lambda \sum_{j=1}^{a-1} h_{a-j}^*(h_a)\, h_j^*(l_{b,...}) \\ &\quad + \lambda \sum_{i>0} (-1)^i \Big[h_a^*(h_{a+i})\, h_i^*(l_{b,...}) + h_{a+i} h_a^* h_i^*(l_{b,...}) \\ &\quad + \lambda \sum_{j=1}^{a-1} h_{a-j}^*(h_{a+i})\, h_j^* h_i^*(l_{b,...}) \Big] \\ &= l_{b,...} + \lambda^2 \sum_{j=1}^{b} h_j h_j^*(l_{b,...}) + \lambda^2 \sum_{i>0} (-1)^i h_i h_i^*(l_{b,...}) \\ &\quad + \lambda^3 \sum_{i>0} \sum_{j=1}^{b} (-1)^i h_{i+j} h_i^* h_j^*(l_{b,...}) \\ &= l_{b,...} + \lambda^2 \sum_{k=1}^{b} h_k \left(h_k^* + (-1)^k h_k^* + \lambda \sum_{i=1}^{k-1} (-1)^i h_i^* h_{k-i}^* \right) (l_{b,...}) \\ &= l_{b,...}, \end{aligned}$$

as required, by the following lemma.

LEMMA. *For all* $k>0$, $\lambda(h_k^* + (-1)^k h_k^* + \lambda \sum_{i=1}^{k-1} (-1)^i h_i^* h_{k-i}^*) = 0$.

The proof is trivial for odd k, and is immediate from (SQ) for even k.

Proof of (D). Proceeding similarly to the proof of (C),

$$
\begin{aligned}
&h^*_{a+n}(l_{a,b,\dots})\\
&\quad= h^*_{a+n}(h_a l_{b,\dots}) + \lambda \sum_{i>0} (-1)^i h^*_{a+n}[(h_{a+i})(h^*_i l_{b,\dots})]\\
&\quad= \lambda h^*_a(h_a)\, h^*_n(l_{b,\dots}) + \sum_{j=n+1}^{b} \lambda h^*_{a+n-j}(h_a)\, h^*_j(l_{b,\dots})\\
&\qquad + \lambda \sum_{i>0} (-1)^i \Big[h^*_{a+n}(h_{a+i})\, h^*_i(l_{b,\dots})\\
&\qquad + \lambda \sum_{j=1}^{b} \lambda h^*_{a+n-j}(h_{a+i})\, h^*_i h^*_j(l_{b,\dots}) \Big]\\
&\quad= \lambda h^*_n(l_{b,\dots}) + \lambda^2 \sum_{j=n+1}^{b} h_{j-n} h^*_j(l_{b,\dots})\\
&\qquad + \lambda(-1)^n h^*_n(l_{b,\dots}) + \lambda^2 \sum_{i=n+1}^{b} (-1)^i h_{i-n} h^*_i(l_{b,\dots})\\
&\qquad + \lambda^2 \sum_{i=1}^{b} \sum_{j=1}^{b} (-1)^i h^*_{a+n-j}(h_{a+i})\, h^*_i h^*_j(l_{b,\dots})\\
&\quad= [1+(-1)^n]\, \lambda h^*_n(l_{b,\dots}) + \lambda^2 \sum_{k=n+1}^{b} [1+(-1)^k]\, h_{k-n} h^*_k(l_{b,\dots})\\
&\qquad + \lambda^2 \sum_{\substack{i+j=n\\ i\geqslant 1,\, j\geqslant 1}} (-1)^i h^*_i h^*_j(l_{b,\dots}) + \lambda^3 \sum_{\substack{i+j>n\\ i\geqslant 1,\, j\geqslant 1}} (-1)^i h_{i+j-n} h^*_i h^*_j(l_{b,\dots})\\
&\quad= \lambda \Big[h^*_n + (-1)^n h^*_n + \lambda \sum_{i=1}^{n-1} (-1)^i h^*_i h^*_{n-i} \Big] (l_{b,\dots})\\
&\qquad + \lambda \sum_{k>n} h_{k-n} \lambda \Big[h^*_k + (-1)^k h^*_k + \lambda \sum_{i=1}^{k-1} (-1)^i h^*_i h^*_{k-i} \Big] (l_{b,\dots})\\
&\quad= 0 + \lambda \sum_{k>n} h_{k-n} \cdot 0 = 0,
\end{aligned}
$$

using the lemma.

This completes the proof of the theorem. A superficially slicker proof of (C) and (D) would use the same manipulations to first show

$$h^*_a R_a(x) \equiv x \qquad (\text{mod } J)$$

and

$$h^*_{a+n} R_a(x) \equiv 0 \qquad (\text{mod } J) \quad \text{for } n > 0,$$

where J is the ideal generated by $\{h^*_i h^*_j(x) : i \geqslant 0, j \geqslant a\}$.

The coefficients $v_{\alpha\beta} \in L$ in

$$h_\alpha = l_\alpha + \sum_{\beta<\alpha} v_{\alpha\beta} l_\beta$$

are the right side of (E) below in the case when α and β partition the same integer.

$$\text{(E)} \quad h^*_\alpha l_\beta = \sum_{\mathscr{C}} \rho^{q(\mathscr{C})} \lambda^{p(\mathscr{C})} l_{\gamma(\mathscr{C})}$$

$$\text{(F)} \quad l^*_\alpha l_\beta = \sum_{\mathscr{C}'} \rho^{q(\mathscr{C}')} \lambda^{p'(\mathscr{C}')} l_{\gamma(\mathscr{C}')}$$

(notation explained below). Formula (F) is a "Littlewood–Richardson L-rule," closely related to Stembridge's "LR rule for Q_α" [St, 8.3]. The special case when α has length one, $l^*_a l_\beta = h^*_a l_\beta$, common to (E) and (F), includes (C) and (D); gives a quick method to invert the matrix $(v_{\alpha\beta})$ by substitution into the defining formula for l_α; yields easily the proof of (E) in general, by analogy with [Z, proof of 4.14]; and gives the inductive strategy, directly analogous to that explained in [Z, p. 51], for the proof of identities in H such as (F).

To explain the notation in (E) and (F): $\mathscr{C}$ and $\mathscr{C}'$ range over all configurations consisting of the shifted Young diagram of β [St] with certain nodes replaced by integers, satisfying (i), (ii), ...

(i) The number of i's entered is the ith term of α (skew shifted tableau of *content*, or *type*, α), and no other integers are entered;

(ii) the nodes unreplaced by entries are the shifted Young diagram of another partition into distinct parts, denoted $\gamma(\mathscr{C})$ or $\gamma(\mathscr{C}')$ (skew shifted tableau of *shape* $\beta \backslash \gamma(\mathscr{C})$);

(iii) entries are non-decreasing along rows (left to right) and down columns, and are strictly increasing down diagonals.

This describes the $\mathscr{C}$ in (E). The $\mathscr{C}'$ in (F) must also satisfy:

(iv) Entries are *markable* in such a way as to satisfy Stembridge's lattice property [St, 8.5], as well as the marking conditions [St, 6.2(2) and (3)].

The exponents of λ in (E) and (F) are defined as follows. Let $a(\mathscr{C})$ be the number of integers which occur as entries but do not occur on the main diagonal; that is,

$$a(\mathscr{C}) = \text{length}(\alpha) - \#(\text{entries on main diagonal}).$$

An entry of $\mathscr{C}$ is *freemarkable* if (i) it is not one the main diagonal; (ii) the entry to the left is smaller if it exists; and (iii) the entry below is larger if it exists. (The remaining off-diagonal entries have no choice between being marked or unmarked for a marking of $\mathscr{C}$ to satisfy [St, 6.2(2) and (3)].) Among the freemarkable entries of $\mathscr{C}'$ there may remain some for which being marked or unmarked will not affect the lattice property [St, 8.5] (for example, the first occurrence of any integer, if off-diagonal, when reading rows left to right starting with the bottom row). Such entries will be called *remarkable*. The exponents of λ in (E) and (F) are then given by

$$p(\mathscr{C}) = 2 \ \# \text{ freemarkable entries} - a(\mathscr{C})$$

$$p'(\mathscr{C}') = 2 \ \# \text{ remarkable entries} - a(\mathscr{C}').$$

Since remarkability can depend on the marks on other entries, one interprets $\lambda^{p'(\mathscr{C}')+a(\mathscr{C}')}$ as a sum of powers of λ^2 which when $\lambda^2=2$ yields the number of marked configuratons which agree with $\mathscr{C}'$ if marks are ignored.

The exponent $q(\mathscr{C})$ or $q(\mathscr{C}')$ of ρ is of no interest when $p(\mathscr{C})$ or $p'(\mathscr{C}')$ is positive. This is always true unless α is a subsequence of β, and is also true in that case for all $\mathscr{C}$ except one. When α is a subsequence of β there is a unique $\mathscr{C}$ for which $\gamma(\mathscr{C})$ is the complementary subsequence. It also occurs as a $\mathscr{C}'$ in formula (F). For this $\mathscr{C}$, $p(\mathscr{C}) = p'(\mathscr{C}) = 0$, and $q(\mathscr{C})$ (of minor interest mod 2) is

$$q(\mathscr{C}) = \sum_{j=1}^{t} (i_j - j)(b_{i_j} + 1) + (t - j - 1)(b_{i_{j-1}+1} + \cdots + b_{i_j - 2} + b_{i_j - 1}),$$

where $\beta = (b_1, b_2, ...)$ and $\alpha = (b_{i_1}, ..., b_{i_t})$.

For readers unfamiliar with the PSH-point of view, we should justify the Littlewood–Richardson designation of (F) by noting that it is trivially equivalent to each of the following formulae for the coproduct and product in H:

$$\Delta l_\beta = \sum_{\alpha} \sum_{\mathscr{C}' \text{ in (F)}} \rho^{q(\mathscr{C}')} \lambda^{p'(\mathscr{C}')} l_\alpha \otimes l_{\gamma(\mathscr{C}')}$$

$$l_\alpha l_\gamma = \sum_{\beta} \sum_{\substack{\mathscr{C}' \text{ as in (F)} \\ \gamma(\mathscr{C}') = \gamma}} \rho^{q(\mathscr{C}')} \lambda^{p'(\mathscr{C}')} l_\beta.$$

References

[BH] M. Bean and P. Hoffman, Zelevinski algebras related to projective representations, *Trans. Amer. Math. Soc.* **309** (1988), 99–111.

[HH1] P. N. Hoffman and J. F. Humphreys, Hopf algebras and projective representations of $G \wr S_n$ and $G \wr A_n$, *Canad. J. Math.* **38**(6) (1986), 1380–1458.

[HH2] P. Hoffman and J. Humphreys, Primitives in the Hopf algebra of projective S_n-representations, *J. Pure Appl. Algebra* **47** (1987), 155–164.

[Sa] B. E. Sagan, Shifted tableaux, Schur Q-functions and a conjecture of R. Stanley, *J. Combin. Theory Ser. A* **45** (1987), 62–103.

[Sc] I. Schur, Über die Darstellung der symmetrischen und der alterierenden Gruppe durch gebrochene lineare Substitutionen, *in* "Collected Works," Vol. I, pp. 346–441, Springer-Verlag, Berlin/New York, 1973.

[St] J. R. Stembridge, Shifted tableaux and projective representations of symmetric groups, *Adv. in Math.* **74** (1989), 87–134.

[W] D. R. Worley, "A Theory of Shifted Young Tableaux," Ph. D. thesis, MIT, 1984.

[Z] A. V. Zelevinski, "Representations of Finite Classical Groups," SLN 869, Springer, 1981.

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