

ON POWERS IN SHIFTED PRODUCTS

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ABSTRACT. In this note we give an estimate for the size of a subset A of $\{1, \dots, N\}$ which has the property that the product of any two distinct elements of A plus 1 is a perfect power.

1. INTRODUCTION

Let V denote the set of all positive integers which are of the form x^k with x and k integers and k at least 2. Thus V is the set of positive integers which are perfect powers. In [6] Gyarmati, Sárközy and Stewart showed that if N is a positive integer and A is a subset of $\{1, \dots, N\}$ with the property that $aa' + 1$ is in V whenever a and a' are distinct elements of A then $|A|$, the cardinality of A , is not large. In particular, they showed that for N sufficiently large

$$(1.1) \quad |A| \leq 340(\log N)^2 / \log \log N.$$

In addition they conjectured that $|A|$ is bounded by an absolute constant. In [8] Luca showed that this follows as a consequence of the *abc* conjecture. Further he improved on (1.1) by showing that there is a positive number c_0 such that for N sufficiently large

$$(1.2) \quad |A| < c_0(\log N / \log \log N)^{3/2}.$$

Estimate (1.1) was proved by combining results from extremal graph theory with a gap principle due to Gyarmati [5] which allows one to push apart integers whose shifted product is a fixed power. The improvement (1.2) of Luca was due to his more efficient treatment of the large powers which might

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occur. He introduced estimates for linear forms in the logarithms of algebraic numbers into his argument to this end. The linear forms Luca considers consist of 4 terms. The purpose of this note is to show that a further improvement of (1.2) is possible by a modification of Luca's argument which allows one to deal with linear forms in only 2 terms. We shall prove the following result.

THEOREM 1.1. *There exists an effectively computable positive number c_1 such that if N is a positive integer with $N \geq 2$ and A is a subset of $\{1, \dots, N\}$ with the property that $aa' + 1$ is a perfect power whenever a and a' are distinct integers from A then*

$$|A| < c_1 \log N.$$

2. PRELIMINARY LEMMAS

LEMMA 2.1. *There is no set of six positive integers $\{a_1, \dots, a_6\}$ with the property that $a_i a_j + 1$ is a square for $1 \leq i < j \leq 6$.*

PROOF. This is [4, Theorem 2]. □

LEMMA 2.2. *Let n and r be integers with $3 \leq r \leq n$. Let G be a graph on n vertices with at least*

$$\frac{r-2}{2(r-1)} n^2$$

edges. Then G contains a complete subgraph on r edges.

PROOF. This follows from Turán's graph theorem, see [9] or [3, Lemma 3]. □

LEMMA 2.3. *Let G be a graph with $n (> 1)$ vertices and e edges and suppose that*

$$e > \frac{1}{2}(n^{3/2} + n - n^{1/2}).$$

Then G contains a cycle of length 4.

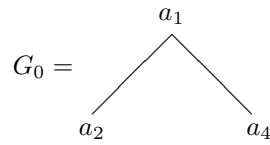
PROOF. This is a special case of [2, Theorem 2.3, Chapter VI] and is due to Kővári, Sós and Turán [7]. □

We shall need an extension of Lemma 2.3 to the case when G is a graph of k colours and the cycle of length 4 is coloured in a certain way.

LEMMA 2.4. *Let G be a graph with n vertices and e edges with the edges coloured by k colours. Suppose that G does not contain a cycle through vertices a_1, a_2, a_3, a_4 where the edges from a_1 to a_2 and from a_1 to a_4 have the same colour and where the edges from a_2 to a_3 and from a_3 to a_4 have the same colour. Then*

$$e \leq k^{1/2} n^{3/2} + kn.$$

PROOF. We will count the number of subgraphs G_0 of G of the form



where the edges (a_1, a_2) and (a_1, a_4) are coloured by the same colour. Let the degree of a_i coloured by the j -th colour be $d_{i,j}$. Then the number of subgraphs G_0 is exactly

$$\sum_{i=1}^n \sum_{j=1}^k \binom{d_{i,j}}{2}.$$

On the other hand this number is less or equal to $\binom{n}{2}$ since for every pair (a_2, a_4) there exists at most one a_1 such that the edges (a_1, a_2) and (a_1, a_4) have the same colour. Thus

$$\sum_{i=1}^n \sum_{j=1}^k \binom{d_{i,j}}{2} \leq \binom{n}{2}.$$

Since $\sum_{i=1}^n \sum_{j=1}^k d_{i,j} = 2e$ we get

$$\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^k d_{i,j}^2 - e \leq \frac{n(n-1)}{2}.$$

By the Cauchy-Schwarz inequality

$$\frac{\left(\sum_{i=1}^n \sum_{j=1}^k d_{i,j}\right)^2}{2kn} - e \leq \frac{n(n-1)}{2}$$

and so

$$\frac{2e^2}{kn} - e \leq \frac{n(n-1)}{2}.$$

Thus

$$e \leq ((4kn^2(n-1) + k^2n^2)^{1/2} + kn)/4$$

and the result now follows. \square

LEMMA 2.5. *Let k be an integer with $k \geq 2$ and let a_1, a_2, a_3 and a_4 be positive integers with $a_1 < a_3$ and $a_2 < a_4$. If $a_1a_2 + 1, a_1a_4 + 1, a_2a_3 + 1$ and $a_3a_4 + 1$ are k -th powers, then*

$$a_3a_4 > (a_1a_2)^{k-1}.$$

PROOF. This follows from the proof of [5, Theorem 1]. \square

For any non-zero rational number α , where $\alpha = a/b$ with a and b coprime integers, we put $H(\alpha) = \max\{|a|, |b|\}$.

LEMMA 2.6. *Let b_1 and b_2 be non-zero integers and let α_1 and α_2 be non-zero rational numbers. Put $A_i = \max\{2, H(\alpha_i)\}$ for $i = 1, 2$, $B = \max\{|b_1|, |b_2|, 2\}$ and $\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2$ where the logarithms take their principal values. There exists an effectively computable positive constant C such that if $\Lambda \neq 0$ then*

$$|\Lambda| > \exp(-C \log A_1 \log A_2 \log B).$$

PROOF. This follows from the Main Theorem of [1]. □

3. PROOF OF THEOREM 1.1

Let A be a subset of $\{1, \dots, N\}$ with the property that $aa' + 1$ is in V whenever a and a' are distinct integers from A . We may suppose that

$$(3.1) \quad |A| > \log N,$$

since otherwise our result holds. Let c_1, c_2, \dots denote effectively computable positive numbers. We shall suppose that N is sufficiently large that

$$(3.2) \quad (\log N)/2 \log \log N > 16.$$

Notice that there is an integer m with

$$1 \leq m \leq \frac{\log((\log N)/\log 2)}{\log 2},$$

such that A has more than $(|A| - 3)/((\log((\log N)/\log 2))/\log 2)$ elements from $\{2^{2^m}, 2^{2^m} + 1, \dots, 2^{2^{m+1}} - 1\}$. We shall denote these elements by A_m and put $n = |A_m|$ and $M = 2^{2^{m+1}}$. Then, for $N > c_1$,

$$(3.3) \quad n > \frac{|A|}{2 \log \log N}.$$

Further, by (3.1), (3.2) and (3.3),

$$(3.4) \quad M > 16.$$

Form the complete graph G whose vertices are the elements of A_m . G has $\binom{n}{2}$ edges and for each pair (a, a') of vertices of G we colour the edge between a and a' with the smallest prime p for which $aa' + 1$ is a perfect p -th power.

By Lemma 2.2, if the number of edges of G with the colour 2 exceeds $(2/5)n^2$ then there is a complete subgraph of G on 6 vertices coloured with 2 and this is impossible by Lemma 2.1. Therefore the number of edges of G with a colour different from 2 is at least $\binom{n}{2} - (2/5)n^2 = (n^2/10) - (n/2)$.

Put

$$(3.5) \quad t = (9C \log M \log \log M)^{1/2},$$

where C is the positive number which occurs in Lemma 2.6. Let G_1 be the subgraph of G consisting of the vertices of G together with the edges of G which are coloured with a prime p for which

$$(3.6) \quad 3 \leq p \leq t$$

and let G_2 be the subgraph of G consisting of the vertices of G together with the edges of G which are coloured with a prime p for which

$$(3.7) \quad t < p < (2 \log M) / \log 2.$$

Suppose that G_1 contains at least $(n^2/20) - (n/2)$ edges. The number of colours of G_1 is $\pi(t) - 1$ and, by the prime number theorem and (3.5), this is at most $c_2((\log M) / \log \log M)^{1/2}$. Thus there is a colour of G_1 which occurs on at least $((n^2/20) - (n/2)) / c_2((\log M) / \log \log M)^{1/2}$ different edges. Since $M \leq N^2$ we see from (3.3) that if

$$(3.8) \quad |A| > c_3 \log N,$$

then there is a colour associated with more than $(n^{3/2} + n - n^{1/2})/2$ edges. Therefore, by Lemma 2.3, G_1 contains a monochromatic cycle of length 4. In particular, there exist integers a_1, a_2, a_3 and a_4 from A_m and a prime p satisfying (3.6) for which $a_1a_2 + 1, a_2a_3 + 1, a_3a_4 + 1$ and $a_1a_4 + 1$ are p -th powers. Without loss of generality one may suppose that $a_1 < a_3$ and $a_2 < a_4$. Thus, by Lemma 2.5,

$$(3.9) \quad a_3a_4 > (a_1a_2)^2.$$

But a_1, a_2, a_3 and a_4 are in $\{2^{2^m}, \dots, 2^{2^{m+1}} - 1\}$ and so

$$a_3a_4 < 2^{2^{m+2}} \leq (a_1a_2)^2,$$

which contradicts (3.9). Accordingly either (3.8) is false, in which case our result follows, or G_1 has fewer than $(n^2/20) - (n/2)$ edges. We may assume the latter possibility and so G_2 has at least $n^2/20$ edges.

It follows from (3.4), (3.7) and the prime number theorem that the number of colours of G_2 is at most $c_4(\log M) / \log \log M$. Therefore since $N^2 \geq M$ and (3.3) holds, if $|A|$ exceeds $c_5 \log N$ then by Lemma 2.4, G_2 contains a cycle through vertices a_1, a_2, a_3 and a_4 for which the edge between a_1 and a_2 and the edge between a_1 and a_4 have the same colour and the edge between a_2 and a_3 and the edge between a_3 and a_4 have the same colour. In particular, there exist primes p_1 and p_2 in the range given by (3.7) and integers x_1, x_2, x_3 and x_4 for which

$$\begin{aligned} a_1a_2 + 1 &= x_1^{p_1}, & a_2a_3 + 1 &= x_2^{p_2}, \\ a_3a_4 + 1 &= x_3^{p_2}, & a_4a_1 + 1 &= x_4^{p_1}. \end{aligned}$$

We observe, as in [8, Lemma 3.1], that

$$(x_1^{p_1} - 1)(x_3^{p_2} - 1) = (x_2^{p_2} - 1)(x_4^{p_1} - 1),$$

hence

$$(3.10) \quad x_1^{p_1} x_3^{p_2} - x_2^{p_2} x_4^{p_1} = x_1^{p_1} + x_3^{p_2} - x_2^{p_2} - x_4^{p_1}.$$

Since $x_1^{p_1} + x_3^{p_2} - x_2^{p_2} - x_4^{p_1} = (a_1 - a_3)(a_2 - a_4)$ and since the a_i 's are distinct we see that

$$x_1^{-p_1} x_3^{-p_2} x_2^{p_2} x_4^{p_1} \neq 1.$$

Thus, if we put

$$(3.11) \quad \Lambda = p_1 \log(x_4/x_1) + p_2 \log(x_2/x_3)$$

we see that $\Lambda \neq 0$. We may assume, without loss of generality, that

$$x_1^{p_1} = \max\{x_1^{p_1}, x_2^{p_2}, x_3^{p_2}, x_4^{p_1}\}.$$

Therefore, by (3.10),

$$(3.12) \quad \left| \frac{x_2^{p_2} x_4^{p_1}}{x_1^{p_1} x_3^{p_2}} - 1 \right| \leq \frac{2}{x_3^{p_2}}.$$

Since a_3 and a_4 are at least $M^{1/2}$ in size

$$x_3^{p_2} > M,$$

and so, by (3.11) and (3.12),

$$|e^\Lambda - 1| < \frac{2}{M}.$$

Observe that if y is a real number and $|e^y - 1| < 1/8$ then $|y| < 1/2$. Further $|e^y - 1| \geq |y|/2$ for $|y| < 1/2$ and so, since $M \geq 16$,

$$|\Lambda| < \frac{4}{M},$$

whence

$$(3.13) \quad \log |\Lambda| < -\frac{1}{2} \log M.$$

We now apply Lemma 2.6 with $\alpha_1 = x_1/x_4$, $\alpha_2 = x_2/x_3$ and $B = \max(p_1, p_2, 2)$. Note that, for $i = 1, 2$,

$$\log H(\alpha_i) \leq (2 \log M)/t.$$

By Lemma 2.6,

$$\log |\Lambda| > -4C((\log M)/t)^2 \log \log M,$$

and so, by (3.13),

$$t^2 < 8C \log M \log \log M.$$

However, this contradicts our choice of t in (3.5). Accordingly $|A|$ is less than $c_5 \log N$ and the result follows.

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REFERENCES

- [1] A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. reine angew. Math. **442** (1993), 19-62.
- [2] B. Bollobás, *Extremal Graph Theory*, London Mathematical Society Monographs No. 11, Academic Press, London, New York, San Francisco, 1978.
- [3] Y. Bugeaud and K. Gyarmati, *On generalizations of a problem of Diophantus*, Illinois J. Math. **48** (2004), 1105-1115.
- [4] A. Dujella, *There are only finitely many Diophantine quintuples*, J. reine angew. Math. **566** (2004), 183-214.
- [5] K. Gyarmati, *On a problem of Diophantus*, Acta Arith. **97** (2001), 53-65.
- [6] K. Gyarmati, A. Sárközy and C.L. Stewart, *On shifted products which are powers*, Mathematika **49** (2002), 227-230.
- [7] T. Kövári, V. Sós and P. Turán, *On a problem of K. Zarankiewicz*, Colloq. Math. **3** (1954), 50-57.
- [8] F. Luca, *On shifted products which are powers*, Glas. Mat. Ser. III **40** (2005), 13-20.
- [9] P. Turán, *On an extremal problem in graph theory*, Mat. Fiz. Lapok **48** (1941), 436-452 (in Hungarian).

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