Studies in Pure Mathematics
To the Memory of Paul Turán

On density-difference sets of sets of integers

by

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Abstract

This note is a continuation of our paper [4]. We prove that density-difference sets do not have the superset property, but that both the union and the intersection of any two density-difference sets is a density-difference set. Finally we show that by repeating the operation of forming the ordinary-difference set we obtain an arithmetical progression after a finite number of steps.

1. Introduction

Let $\mathbf{N}_0$ denote the non-negative integers and let A be any subset of $\mathbf{N}_0$. We denote the number of elements of A by $|A|$ and the number of elements of A which are less than x by $|A|_x$. As usual we define the upper density of A by $\bar{d}(A) = \limsup_{x\to\infty} |A|_x/x$ and the lower density of A by $\underline{d}(A) = \liminf_{x\to\infty} |A|_x/x$. If $\bar{d}(A) = \underline{d}(A)$, then we define this limit value to be the density $d(A)$ of A.

Let $d \in \mathbf{Z}$. By $A - d$ we denote the set of integers $b \in \mathbf{N}_0$ with $b + d \in A$. For the sake of brevity we write $A[d]$ instead of $A \cap (A - d)$. We define the ordinary-difference set $\mathscr{D}(A)$ of A by

$$\mathscr{D}(A) = \{d \in \mathbf{N}_0 \mid A[d] \neq \emptyset\},$$

the infinite-difference set $\mathscr{D}_\infty(A)$ of A by

$$\mathscr{D}_\infty(A) = \{d \in \mathbf{N}_0 \mid |A[d]| = \infty\}$$

and the density-difference set $\mathscr{D}_0(A)$ of A by

$$\mathscr{D}_0(A) = \{d \in \mathbf{N}_0 \mid \bar{d}(A[d]) > 0\}.$$

In the last few years difference sets have been investigated by several people. See [5], or for recent results not mentioned in this survey paper [1], [2], [3]. Most authors have considered only ordinary-difference sets. Their results, however, can often be extended

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to the other types of difference sets. Our first theorem, which is an extension of Theorem 5 of [4], is sometimes useful in this connection.

Theorem 1. *Given a set $A \subseteq \mathbf{N}_0$ there exists a set $B \subseteq \mathbf{N}_0$ with $\underline{d}(B) \geqq \bar{d}(A)$ such that $\mathscr{D}(B) \subseteq \mathscr{D}_0(A)$.*

In [4] we proved that the collection of infinite-difference sets associated with sets of positive upper density is a filter of the set of all subsets of $\mathbf{N}_0$. In particular this implies that if A has positive upper density and $\mathscr{D}_\infty(A) \subseteq K \subseteq \mathbf{N}_0$, then there exists a set B with positive upper density such that $\mathscr{D}_\infty(B) = K$. The analogous property does not hold for the other two types of difference sets. Let E be the set of non-negative even integers. We have $\mathscr{D}(E) = E$, but it is easy to see that $E \cup \{1\}$ is not the ordinary-difference set of any set. Similarly we have $\mathscr{D}_0(E) = E$.

Theorem 2. *There is no set A such that $\mathscr{D}_0(A) = E \cup \{1\}$.*

Another consequence of the filter property is that the union of two infinite-difference sets of sets of positive upper density is also an infinite-difference set of a set of positive upper density. The analogous assertion does not hold for ordinary-difference sets. Take $A = \{0, 1, 11, 21, 31, \dots\}$ and $B = \{0, 3, 13, 23, 33, \dots\}$. Then $\mathscr{D}(A) \cup \mathscr{D}(B) = \{c \mid c \equiv 0, 1 \text{ or } 3 \pmod{10}\}$ but this is not an ordinary-difference set. However, it is true that the union of any two density-difference sets is a difference set.

Theorem 3. *If A and B are subsets of $\mathbf{N}_0$, then there exists a set $C \subseteq \mathbf{N}_0$ such that $\mathscr{D}_0(C) = \mathscr{D}_0(A) \cup \mathscr{D}_0(B)$.*

A further consequence of the filter property is that the intersection of two infinite-difference sets associated with sets of positive upper density is also an infinite-difference set of a set of positive upper density. While the corresponding assertion does not hold for ordinary-difference sets (see [4]), it does hold for density-difference sets.

Theorem 4. *If A and B are subsets of $\mathbf{N}_0$, then there exists a set $C \subseteq \mathbf{N}_0$ such that $\mathscr{D}_0(C) = \mathscr{D}_0(A) \cap \mathscr{D}_0(B)$. Moreover, $\bar{d}(C[d]) \geqq \bar{d}(A[d]) \cdot \bar{d}(B[d])$ for every $d \in \mathbf{N}_0$.*

On taking $d = 0$ we obtain $\bar{d}(C) \geqq \bar{d}(A) \cdot \bar{d}(B)$. Hence, by Theorem 1 with $C = A$ and some simple arguments, there exists a set C' such that

$$\underline{d}(\mathscr{D}_0(A) \cap \mathscr{D}_0(B)) = \underline{d}(\mathscr{D}_0(C)) \geqq \underline{d}(\mathscr{D}(C')) \geqq \underline{d}(C') \geqq \bar{d}(C) = \bar{d}(A) \cdot \bar{d}(B).$$

So, by repeated application of Theorem 4, we find that for any sets $A_1, A_2, \dots, A_h \subseteq \mathbf{N}_0$,

$$\underline{d}\left(\bigcap_{j=1}^{h} \mathscr{D}_0(A_j) \right) \geqq \prod_{j=1}^{h} \bar{d}(A_j). \tag{1}$$

In [4] we showed that this inequality cannot be improved. Inequality (1) is an improvement of [4] Theorem 1 and it was proved recently by Y. KATZNELSON and by I. Z. RUZSA [3]. It follows from Theorem 4 that there even exists a set C with $\bar{d}(C) \geq \prod_{j=1}^{h} \bar{d}(A_j)$ such that $\mathscr{D}_0(C) = \bigcap_{j=1}^{h} \mathscr{D}_0(A_j)$.

Up to now no simple characterization of any of the three sets of difference sets associated with sets of positive upper density has been found. Our last result, by contrast, demonstrates that if we iterate the operation of taking the ordinary-difference set, a simple stable set occurs after relatively few steps. Put $\mathscr{D}^1(A) = \mathscr{D}(A)$ and $\mathscr{D}^k(A) = \mathscr{D}(\mathscr{D}^{k-1}(A))$ for $k = 2, 3, \ldots$.

Theorem 5. *Let A have positive upper density ε. Then there exists an integer k with $1 \leq k \leq \varepsilon^{-1}$ such that $\mathscr{D}^r(A) = \{jk\}_{j=0}^{\infty}$ for all integers r with $r > 2[(\log \varepsilon^{-1})/\log 2]$.*

The example of the set of integers A_l with $A_l = \{a \equiv 0 \text{ or } 1 \pmod{l}\}$ shows that the lower bound for r cannot be replaced by $[(\log \varepsilon^{-1})/\log 2]$. We wonder whether the assertion of Theorem 5 remains valid if we replace the last inequality by $r > [(\log \varepsilon^{-1})/\log 2] + 1$.

2. Proof of Theorem 1

Put $\varepsilon = \bar{d}(A)$ and let $n \in \mathbf{N}$. Note that we may assume that ε is positive, since otherwise the theorem plainly holds. We prove first that there exists a set

$$R = \{r_j\}_{j=1}^{\infty} \subseteq \mathbf{N}_0$$

with positive upper density such that

$$|A - r_j|_k \geq \varepsilon k \quad \text{for } k = 1, \ldots, n \quad \text{and} \quad j = 1, 2, \ldots.$$

Suppose this statement is false. Put

$$\varepsilon' = \left\{ \max \frac{i}{k} \,\middle|\, i, k \in \mathbf{N}_0,\ 1 \leq k \leq n,\ \frac{i}{k} < \varepsilon \right\}.$$

Note that $\varepsilon' < \varepsilon$ and that for every m except for a set $M^* \subseteq \mathbf{N}_0$ of density zero we have $|A - m|_{k_m} \leq \varepsilon' k_m$ for some k_m with $1 \leq k_m \leq n$. Put $k_m = 1$ if $m \in M^*$. Define the set M inductively by $m_1 = 1$, $m_{j+1} = m_j + k_{m_j}$ for $j = 1, 2, \ldots$. Let $x \in \mathbf{N}$ and define J by the inequalities $m_J \leq x < m_{J+1}$. It follows that

$$|A|_x \leq \varepsilon' m_J + x - m_J + |M^*|_x \leq \varepsilon' x + n + |M^*|_x.$$

Since $\bar{d}(M^*) = 0$, it follows that $\bar{d}(A) \leq \varepsilon'$, which is a contradiction.

Let $\{r_j^{(n)}\}_{j=1}^{\infty}$ be a set of positive upper density such that $|A - r_j^{(n)}|_k \geq \varepsilon k$ for $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots$. Denote $\{0, 1, \ldots, n-1\}$ by $\hat{n}$. We consider the sets $(A - r_j^{(n)}) \cap \hat{n}$. By the pigeon hole principle there exists a subset $\{s_j^{(n)}\}_{j=1}^{\infty}$ of $\{r_j^{(n)}\}_{j=1}^{\infty}$ of positive upper density such that $(A-s_j^{(n)})\cap\hat{n}$ is the same set $S^{(n)}$ for every j. We obtain in this way a set $S^{(n)}$, for every positive integer n, such that for $k=1,2,\ldots,n$ the number of elements less than k is at least εk. We now construct the set B by induction. Suppose $B\cap\hat{n}$ has been constructed in such a way that there are infinitely many integers v with $S^{(v)}\cap\hat{n}=B\cap\hat{n}$. We put $n\in B$ if and only if there are infinitely many integers v' among these integers v with $n\in S^{(v')}$. It follows that there are infinitely many integers v with $S^{(v)}\cap\widehat{n+1}=$ $=B\cap\widehat{n+1}$. By construction, for some $v>n$,

$$|B|_n=|S^{(v)}|_n\geqq\varepsilon n\,.$$

Thus $\underline{d}(B)\geqq\varepsilon=\bar{d}(A)$.

Let $d\in\mathscr{D}(B)$. Then $n\in B[d]$ for some n and hence $n\in S^{(v)}[d]$ for some integer v. Therefore $n+s_j^{(v)}\in A$ and $n+d+s_j^{(v)}\in A$ for a set $\{s_j^{(v)}\}$ of positive upper density. Thus $d\in\mathscr{D}_0(A)$. This completes the proof.

Note that we have even proved that the Schnirelmann density of $B+1$ is at least ε, since $|B|_n\geqq\varepsilon n$ for every positive integer n.

3. Proof of Theorem 2

Suppose that A is a set for which 1 is an element of $\mathscr{D}_0(A)$. Thus $\bar{d}(A[1])=\varepsilon>0$. We shall show that $\mathscr{D}_0(A)$ contains an odd integer less than $8\varepsilon^{-1}+1$ and larger than 1. This will establish the result. Let $a_1, a_2, \ldots$ be the elements of $A[1]$ in increasing order. We have $\liminf\limits_{j\to\infty} a_j/j=\varepsilon^{-1}$. Hence there exist infinitely many integers a_N for which $a_{N+2}/N<2\varepsilon^{-1}$. Now since

$$\frac{1}{N}\sum_{j=1}^{N}(a_{j+2}-a_j)\leqq\frac{2a_{N+2}}{N}<\frac{4}{\varepsilon},$$

at least one half of the integers a_j with $j\leqq N$ satisfy $a_{j+2}-a_j<8\varepsilon^{-1}$. Thus $a_{j+2}-a_j=d$ for some integer d with $1<d<8\varepsilon^{-1}$ for a set of integers a_j of positive upper density. Since $a_{j+2}+1\in A$ for every j, both d and $d+1$ are in $\mathscr{D}_0(A)$ and thus there is an odd integer different from 1 in $\mathscr{D}_0(A)$ as required.

4. Proof of Theorem 3

We may assume that the upper density of both A and B is positive. Define C by $j\in C$ if and only if

$$\begin{cases} j-2^{2i}\in A & \text{for} \quad 2^{2i}\leqq j<2^{2i+1}, \\ j-2^{2i+1}\in B & \text{for} \quad 2^{2i+1}\leqq j<2^{2i+2}, \end{cases} \qquad i=0,1,2,\ldots.$$

Suppose that n is a positive integer such that $|A|_n > n\bar{d}(A)/2$. Take i such that $2^{2i-2} \leqq n < 2^{2i}$. Then

$$|C|_{2^{2i+1}} \geqq |A|_{2^{2i}} \geqq \frac{n\bar{d}(A)}{2} \geqq \frac{2^{2i+1}\bar{d}(A)}{16}.$$

Thus $\bar{d}(C) \geqq \bar{d}(A)/16 > 0$.

Let $d \in \mathscr{D}_0(A)$. Let $\varepsilon > 0$ be such that $|A[d]|_{n-d} > \varepsilon n$ for infinitely many n. For such an n define the integer i by $2^{2i-2} \leqq n < 2^{2i}$. Then

$$|C[d]|_{2^{2i+1}} \geqq |A[d]|_{2^{2i}-d} \geqq \varepsilon n \geqq \frac{\varepsilon \cdot 2^{2i+1}}{8}.$$

Hence, $d \in \mathscr{D}_0(C)$. Thus $\mathscr{D}_0(A) \subseteq \mathscr{D}_0(C)$. Similarly, $\mathscr{D}_0(B) \subseteq \mathscr{D}_0(C)$.

Finally, suppose $d \notin \mathscr{D}_0(A) \cup \mathscr{D}_0(B)$. Let $0 < \varepsilon < 1$. Take n_0 so large that both $|A[d]|_n < \varepsilon n$ and $|B[d]|_n < \varepsilon n$ for $n \geqq n_0$. Let n and i be integers with $2^{2i} > n \geqq 2^{2i-2} \geqq n_0$. Then

$$|C[d]|_n \leqq \sum_{j=0}^{i-1} |A[d]|_{2^{2j}} + \sum_{j=0}^{i-1} |B[d]|_{2^{2j+1}} + 2id \leqq$$

$$\leqq \varepsilon \sum_{j=0}^{i-1} 2^{2j} + \varepsilon \sum_{j=0}^{i-1} 2^{2j+1} + c + 2id,$$

where c is some constant. Hence,

$$|C[d]|_n \leqq \varepsilon \cdot 2^{2i} + c + 2id \leqq 4\varepsilon n + c + d\frac{\log 4n}{\log 2}.$$

Thus $\bar{d}(C[d]) \leqq 4\varepsilon$. Since ε was arbitrary, $d \notin \mathscr{D}_0(C)$. This completes the proof of the theorem.

5. Proof of Theorem 4

For any positive integer n we denote the set $\{0, 1, \ldots, n-1\}$ by $\hat{n}$. Let $\kappa, \lambda \in \mathbf{N}_0$ with $\kappa > \lambda > 0$. Put $\tau = [\log \lambda]$ and $\sigma = (\kappa + \lambda)(\lambda + \tau)$. We define a subset $S = S_{\kappa,\lambda}$ of $\hat{\sigma}$ such that

$$(2) \qquad |S[d]| = |A[d]|_{\kappa-d} \cdot |B[d]|_{\lambda-d}$$

for every $d \in \hat{\tau}$. Namely, if $x = n(\lambda+\tau)+j$ with $n \in \widehat{\kappa+\lambda}, j \in \widehat{\lambda+\tau}$, then $x \in S$ if and only if $j \in ((A \cap \hat{\kappa}) - n + \lambda - 1) \cap B \cap \hat{\lambda}$. In the next paragraph we show that (2) holds.

Assume $d \in \hat{\tau}$, $a \in A[d] \cap \widehat{\kappa - d}$, $b \in B[d] \cap \widehat{\lambda - d}$. Then $a, a+d \in A \cap \hat{\kappa}$ and b, $b+d \in B \cap \hat{\lambda}$. Put $n = a - b + \lambda - 1$. It follows that $n(\lambda+\tau)+b \in S$ and $n(\lambda+\tau)+b+d \in S$.

Hence $n(\lambda+\tau)+b\in S[d]$. Since different pairs a, b lead to different numbers $n(\lambda+\tau)+b$, this proves that

$$|A[d]|_{\kappa-d}\cdot|B[d]|_{\lambda-d}\leqq|S[d]|\,.$$

On the other hand, assume $n(\lambda+\tau)\hat{+}j\in S[d]$ with $n\in\widehat{\kappa+\lambda}$, $j\in\widehat{\lambda+\tau}$, $d\in\hat{\tau}$. Then both $n(\lambda+\tau)+j$ and $n(\lambda+\tau)+j+d$ belong to S. Hence $j<\lambda$, $j+d<\lambda+\tau$ and it follows that $j+d<\lambda$. Further we have $j,j+d\in(A\cap\hat{\kappa})-n+\lambda-1$ and $j,j+d\in B$. Put $a=$ $=j+n-\lambda+1$. We now also have $a,a+d\in A\cap\hat{\kappa}$. Thus $a\in A[d]\cap\widehat{\kappa-d}$ and $j\in B[d]\cap\widehat{\lambda-d}$. Since different numbers $n(\lambda+\tau)+j\in S[d]$ lead to different pairs a,j, we have proved that

$$|S[d]|\leqq|A[d]|_{\kappa-d}\cdot|B[d]|_{\lambda-d}\,.$$

This completes the proof of (2).

Put $\alpha_d=\bar{d}(A[d])$ and $\beta_d=\bar{d}(B[d])$ for $d\in\mathbf{N}_0$. In particular $\alpha_0=\bar{d}(A)$, $\beta_0=\bar{d}(B)$. Let $\{k_j^{(d)}\}_{j=1}^{\infty}$ and $\{l_j^{(d)}\}_{j=1}^{\infty}$ be strictly increasing sequences of positive integers such that

$$\text{(3)}\qquad |A[d]|_{k_j^{(d)}-d}\geqq\left(1-\frac{1}{j}\right)k_j^{(d)}\alpha_d\quad\text{and}\quad|B[d]|_{l_j^{(d)}-d}\geqq\left(1-\frac{1}{j}\right)l_j^{(d)}\beta_d$$

for $j=1,2,3,\ldots$. Next we define two sequences $\{\kappa(h)\}_{h=0}^{\infty}$ and $\{\lambda(h)\}_{h=0}^{\infty}$ by induction. Put $\kappa(0)=k_1^{(0)}$, $\lambda(0)=l_1^{(0)}$. If $h=m^2+d$ with $m\in\mathbf{N}$, $0\leqq d\leqq 2m$, then $\lambda(h)$ is a term of $\{l_j^{(d)}\}_{j=m}^{\infty}$ and $\kappa(h)$ is a term of $\{k_j^{(d)}\}_{j=m}^{\infty}$ chosen in such a way that for $h=1,2,\ldots$

$$\text{(4)}\qquad \lambda(h)>\lambda(h-1)\quad\text{and}\quad\kappa(h)>h\{\lambda(h)+\sum_{j=0}^{h-1}(\kappa(j)+\lambda(j))(\lambda(j)+\log\lambda(j)\}\,.$$

Hence $\{\kappa(h)\}_{h=0}^{\infty}$ and $\{\lambda(h)\}_{h=0}^{\infty}$ are strictly increasing sequences of positive integers. Put $\tau(j)=[\log\lambda(j)]$ and $\sigma(j)=(\kappa(j)+\lambda(j))(\lambda(j)+\tau(j))$ for $j=0,1,2,\ldots$ and $M(-1)=0$ and $M(h)=\sum_{j=0}^{h}\sigma_j$ for $h=0,1,2,\ldots$. We define C by

$$\text{(5)}\qquad (C-M(h-1))\cap\widehat{\sigma(h)}=S_{\kappa(h),\lambda(h)}$$

for $h=0,1,2,\ldots$.

We prove first that

$$\bar{d}(C[d])\geqq\bar{d}(A[d])\cdot\bar{d}(B[d])$$

for $d\in\mathbf{N}_0$. We observe that, by (5), (2) and (3), for $0\leqq d\leqq\tau(m^2)$,

$$|C[d]|_{M(m^2+d)}\geqq|S_{\kappa(m^2+d),\lambda(m^2+d)}[d]|=$$
$$=|A[d]|_{\kappa(m^2+d)-d}\cdot|B[d]|_{\lambda(m^2+d)-d}\geqq$$
$$\geqq\left(1-\frac{1}{m}\right)\kappa(m^2+d)\alpha_d\cdot\left(1-\frac{1}{m}\right)\lambda(m^2+d)\beta_d\,.$$

On the other hand, by (4)

$$M(m^2+d)=\sum_{j=0}^{m^2+d}\sigma(j)\leqq\sigma(m^2+d)+\frac{1}{m}\kappa(m^2+d)\leqq$$

$$\leqq\left\{\left(1+\frac{1}{m^2+d}\right)\left(1+\frac{\log\lambda(m^2+d)}{\lambda(m^2+d)}\right)+\frac{1}{m}\right\}\kappa(m^2+d)\,\lambda(m^2+d).$$

Thus

$$\bar{d}(C[d])\geqq\left(1-\frac{1}{m}\right)^2\left\{\left(1+\frac{1}{m}\right)\left(1+\frac{\log\lambda(m^2+d)}{\lambda(m^2+d)}\right)+\frac{1}{m}\right\}^{-1}\alpha_d\beta_d.$$

If $m\to\infty$, then $\lambda(m^2+d)\to\infty$. Hence, for any $d\in\mathbf{N}_0$,

$$\bar{d}(C[d])\geqq\alpha_d\beta_d=\bar{d}(A[d])\cdot\bar{d}(B[d]).$$

Consequently, $\mathcal{D}_0(A)\cap\mathcal{D}_0(B)\subseteq\mathcal{D}_0(C)$.

The proof of the theorem will be complete after we show that $\mathcal{D}_0(C)\subseteq\mathcal{D}_0(A)\cap\mathcal{D}_0(B)$. Fix some positive integer d with $d\notin\mathcal{D}_0(A)\cap\mathcal{D}_0(B)$. Let $n\in\mathbf{N}$. Take m and y such that

(6) $$n=M(m-1)+y \quad\text{and}\quad y\in\sigma(m).$$

It follows that

(7) $$|C[d]|_n\leqq\sum_{j=0}^{m-1}|S_{\kappa(j),\lambda(j)}[d]|+|S_{\kappa(m),\lambda(m)}[d]|_y.$$

Choose j_0 such that $\tau(j)\geqq d$ for $j\geqq j_0$. We have, by (2), for $j\geqq j_0$,

(8) $$|S_{\kappa(j),\lambda(j)}[d]|\leqq|A[d]|_{\kappa(j)-d}\cdot|B[d]|_{\lambda(j)-d}.$$

Let ε be any positive number. Let c be a constant such that

(9) $$|A[d]|_n<\varepsilon n+c \quad\text{for every}\quad n\in\mathbf{N}$$

if $d\notin\mathcal{D}_0(A)$, and such that

(10) $$|B[d]|_n<\varepsilon n+c \quad\text{for every}\quad n\in\mathbf{N}$$

if $d\notin\mathcal{D}_0(B)$. Then

(11) $$|A[d]|_{\kappa(j)}|B[d]|_{\lambda(j)}<\max\{\varepsilon\kappa(j)+c)\lambda(j),\kappa(j)(\varepsilon\lambda(j)+c)\}\leqq$$
$$\leqq\varepsilon\kappa(j)\lambda(j)+c(\kappa(j)+\lambda(j)).$$

On combining (8) and (11) we obtain

$$\sum_{j=0}^{m-1} |S_{\kappa(j),\lambda(j)}[d]| \leqq \varepsilon \sum_{j=0}^{m-1} \kappa(j)\lambda(j) + c \sum_{j=0}^{m-1} (\kappa(j)+\lambda(j)) + c_1, \tag{12}$$

where c_1 is some constant larger than $\sum_{j=0}^{j_0} |S_{\kappa(j),\lambda(j)}[d]|$.

From now on we assume that $m \geqq j_0$. Put $y=(t-1)(\lambda(m)+\tau(m))+u$ with $0 \leqq u < \lambda(m)+\tau(m)$. If (10) holds, then, by the construction of $S_{\kappa,\lambda}$,

$$\begin{aligned} |S_{\kappa(m),\lambda(m)}[d]|_y &\leqq (t-1)|B[d]|_{\lambda(m)} + |B[d]|_u \leqq \\ &\leqq (t-1)(\varepsilon\lambda(m)+c)+\varepsilon u + c \leqq \varepsilon y + tc \leqq \varepsilon y + (c+\varepsilon)\left(\frac{y}{\lambda(m)}+1\right). \end{aligned} \tag{13}$$

If (9) holds, then, by the construction of $S_{\kappa,\lambda}$,

$$|S_{\kappa(m),\lambda(m)}[d]|_y \leqq \sum_{j=1}^{t} |(A+\lambda(m)-j)[d]|_{\lambda(m)}. \tag{14}$$

The elements of $(A+\lambda(m)-j)[d]$ correspond to elements of $A[d]$. If an element $a \in A[d]$ induces an element of $(A+\lambda(m)-j)[d] \cap \widehat{\lambda(m)}$, then $-\lambda(m)+j \leqq a < j \leqq t$. A fixed number a is counted therefore at most $\min(t, \lambda(m))$ times in the sum. Hence, by (9),

$$\sum_{j=1}^{t} |(A+\lambda(m)-j)[d]|_{\lambda(m)} \leqq \min(t,\lambda(m))|A[d]|_t \leqq \min(t,\lambda(m))(\varepsilon t + c). \tag{15}$$

On combining (14) and (15) we obtain

$$\begin{aligned} |S_{\kappa(m),\lambda(m)}[d]|_y &\leqq \min(t,\lambda(m))(\varepsilon(t-1)+\varepsilon+c) \leqq \\ &\leqq \varepsilon y + (c+\varepsilon)t \leqq \varepsilon y + (c+\varepsilon)\left(\frac{y}{\lambda(m)}+1\right). \end{aligned} \tag{16}$$

From the inequalities (7), (12), (13) and (16), we find for $m \geqq j_0$,

$$|C[d]|_n \leqq \varepsilon\left(y + \sum_{j=0}^{m-1} \kappa(j)\lambda(j)\right) + c\sum_{j=0}^{m-1}(\kappa(j)+\lambda(j)) + c_1 + (c+\varepsilon)\left(\frac{y}{\lambda(m)}+1\right).$$

Hence, by (6) and (4),

$$|C[d]|_n \leqq \varepsilon n + 3c\kappa(m-1) + c_1 + (c+\varepsilon)\left(\frac{n}{\lambda(m)}+1\right).$$

If $n \to \infty$, then $m \to \infty$ and $\lambda(m) \to \infty$. Therefore, since $n \geqq \kappa(m-1)\lambda(m-1)$,

$$\bar{d}(C[d]) \leqq \lim_{\substack{n\to\infty \\ m\to\infty}} \left\{\varepsilon + \frac{3c}{\lambda(m-1)} + \frac{c_1}{n} + (c+\varepsilon)\left(\frac{1}{\lambda(m)} + \frac{1}{n}\right)\right\} = \varepsilon .$$

Since ε was arbitrary, $\bar{d}(C[d])=0$ for every number $d \notin \mathscr{D}_0(A) \cap \mathscr{D}_0(B)$. Thus $\mathscr{D}_0(C) \subseteq \mathscr{D}_0(A) \cap \mathscr{D}_0(B)$. This completes the proof of the theorem.

6. Proof of Theorem 5

We remark that if $\mathscr{D}^r(A) = \mathscr{D}^{r+1}(A)$ then in fact $\mathscr{D}^r(A) = \mathscr{D}^s(A)$ for all $s \geqq r$. We also note that if $0 \in A$ then $A \subseteq \mathscr{D}(A)$ and thus $\mathscr{D}(A) \subseteq \mathscr{D}^2(A) \subseteq \ldots$. We shall prove first that if $\mathscr{D}(A) \neq \mathscr{D}^2(A) \neq \ldots \neq \mathscr{D}^r(A)$, then $r \leqq 2[(\log \varepsilon^{-1})/\log 2]+1$.

Accordingly, let l be an integer which is in $\mathscr{D}^2(A)$ but not in $\mathscr{D}(A)$. We may then write

$$l = (a_1 - a_2) - (a_3 - a_4), \tag{17}$$

where a_1, a_2, a_3, a_4 are integers from A with $a_1 > a_2$ and $a_3 \geqq a_4$. On replacing a_1 in the expression of (17) by any x from A which is larger than a_1 we again find a number which is in $\mathscr{D}^2(A)$. Thus $(A - a_1) + l \subseteq \mathscr{D}^2(A)$. Plainly $\mathscr{D}^2(A)$ also contains the set $A - a_1$. The sets $(A - a_1) + l$ and $A - a_1$ are disjoint, since l is not in $\mathscr{D}(A)$. Thus $\bar{d}(\mathscr{D}^2(A)) \geqq 2\bar{d}(A)$. We may repeat the above argument with A replaced by $\mathscr{D}^{t-2}(A)$ and l replaced by an integer which is in $\mathscr{D}^t(A)$ but not in $\mathscr{D}^{t-1}(A)$, for any integer t with $2 < t \leqq r$.

We then find that $\bar{d}(\mathscr{D}^t(A)) \geqq 2\bar{d}(\mathscr{D}^{t-2}(A))$. In particular,

$$\bar{d}(\mathscr{D}^{2s}(A)) \geqq 2^s \bar{d}(A) = 2^s \varepsilon$$

for $s = [r/2]$. Since the upper density of $\mathscr{D}^{2s}(A)$ is at most 1, we have $s \leqq [(\log \varepsilon^{-1})/\log 2]$ and therefore $r \leqq 2[(\log \varepsilon^{-1})/\log 2]+1$ as was asserted previously.

Thus $\mathscr{D}^s(A) = \mathscr{D}^r(A)$ for all $s \geqq r$ whenever $r > 2[(\log \varepsilon^{-1})/\log 2]$. Put $k\mathbf{N}_0 = \{kt\}_{t=0}^{\infty}$. Plainly if $\mathscr{D}^r(A) = k\mathbf{N}_0$ then $1 \leqq k \leqq \varepsilon^{-1}$, since $\bar{d}(\mathscr{D}^r(A)) \geqq \varepsilon$ for $r \geqq 1$. Therefore to conclude the proof if suffices to show that if $A = \mathscr{D}(A)$ then $A = k\mathbf{N}_0$. Obviously $0 \in A$, so we may write $A = \{0, a_1, a_2, \ldots\}$ with $0 < a_1 < a_2 < \ldots$. Assume $A \neq a_1\mathbf{N}_0$ and let j be the first index for which $a_j \neq ja_1$. Then the difference $a_j - a_1$ is in $\mathscr{D}(A)$ but not in A contradicting the assumption that $A = \mathscr{D}(A)$. Therefore $A = a_1\mathbf{N}_0$ as required. This completes the proof.

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