

On prime factors of sums of integers III

by

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*Dedicated to Professor P. Erdős
on the occasion of his
seventy-fifth birthday*

1. In [1], pp. 36–37, Erdős posed the following questions. (We use a slightly different notation and denote the number of distinct prime divisors of an integer n by $\omega(n)$.) “Let $G(k)$ be a graph of k vertices x_1, \dots, x_k . Let a_1, \dots, a_k be any set of k distinct integers. Put

$$\omega(G(k)) = \min \omega\left(\prod (a_i + a_j)\right)$$

where the factor $a_i + a_j$ occurs if x_i and x_j are joined by an edge and the minimum is extended over all choices of distinct integers a_1, \dots, a_k . It is probably hopeless to determine $\omega(G(k))$ except for very simple graphs, but perhaps one can get conditions on the class of graphs for which $\omega(G(k)) \rightarrow \infty$. One conjecture which just occurs to me states: There is a $g(r) \rightarrow \infty$ as $r \rightarrow \infty$ so that if $G(k)$ has a chromatic number $\geq r$ then $\omega(G(k)) \geq g(r)$. A stronger conjecture: If $G(k)$ has more than kr edges then $\omega(G(k)) \geq g(r)$. Perhaps this is too optimistic”.

We shall show that the latter conjecture is incorrect indeed, but that Theorem 1 implies

$$\omega\left(\prod (a_i + a_j)\right) \gg \log r \quad \text{as } r \rightarrow \infty$$

under the additional assumption that $\omega(a_1 \dots a_k)$ is bounded. Furthermore, it follows from Theorem 2 that if $G(k)$ has more than $k^{3/2}r$ edges or if $G(k)$ contains a complete bipartite subgraph of type $(r, 2)$ then $\omega(G(k)) \gg \log r$ as $r \rightarrow \infty$. Here \gg denotes the Vinogradov symbol. Similar results will also be proved for the products of differences of integers. Our results are applications

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of two deep results (Lemmas 1, 2) which provide good and explicit upper bounds for the numbers of solutions of S -unit equations in two variables.

2. The following example shows that Erdős' stronger conjecture is incorrect. Let k be an odd integer, $k \geq 35$. Put $a_i = i - (k+1)/2$ for $i = 1, \dots, k$. Let p_1, \dots, p_s be the smallest s prime numbers, $s \geq 7$. An integer n is said to be *composed* of p_1, \dots, p_s if n has no prime divisor different from p_1, \dots, p_s . The number, R , of positive integers less than $k/2$ which are composed of p_1, \dots, p_s equals the number of non-negative integer solutions z_1, \dots, z_s of the inequality

$$z_1 \log p_1 + \dots + z_s \log p_s < \log(k/2).$$

Since $p_j \leq 2s \log s$ for $j = 1, \dots, s$ (see [7], formula (3.13)), R is at least the number of non-negative integer solutions z_1, \dots, z_s of the inequality

$$z_1 + \dots + z_s < \frac{\log(k/2)}{\log(2s \log s)},$$

hence

$$R \geq \frac{1}{s!} \left(\frac{\log(k/2)}{\log(2s \log s)} \right)^s \geq 4 \left(\frac{\log(k/2)}{s \log s} \right)^s.$$

For each positive integer n with $1 \leq n < k/2$ there are at least $(k-2)/2$ pairs (i, j) with $1 \leq i < j \leq k$ and $a_i + a_j = \pm n$. Let G be the graph with vertex set $\{x_1, \dots, x_k\}$ such that x_i and x_j are joined by an edge if and only if $a_i + a_j$ is composed of p_1, \dots, p_s . Then the number of edges of G is at least

$$\frac{k-2}{2} R \geq k \left(\frac{\log(k/2)}{s \log s} \right)^s.$$

This implies that the latter conjecture of Erdős is incorrect as we see by taking $r = (\log(k/2)/(s \log s))^s$ and letting $k \rightarrow \infty$, s fixed. It even follows that it is not sufficient to assume that $G(k)$ has more than $k(\log k)^A r$ edges, where A is any constant, to conclude that $\omega(G(k)) \rightarrow \infty$ as $r \rightarrow \infty$.

3. We state a special case of a result of Evertse which shall be used in the proofs of Theorems 1 and 2.

LEMMA 1 (Evertse [2]). *Let v be a non-zero integer. There are at most $3 \times 7^{2s+3}$ pairs of integers x, y each composed of p_1, \dots, p_s such that $x + y = v$.*

The crucial point of the lemma is that the upper bound only depends on s . We used this lemma in [5] to derive a lower bound for the number of

distinct prime divisors in the full product $\prod_{i=1}^k \prod_{j=1}^l (a_i + b_j)$. We refer to [8] for a survey of related results.

4. As to Erdős' second conjecture it is true that if $G(k)$ has more than kr edges then $\omega(\prod (a_i + a_j)) \rightarrow \infty$ as $r \rightarrow \infty$, provided that $\omega(a_1 \dots a_k)$ is bounded. We prove a more general result. We denote the cardinality of a set V by $|V|$.

THEOREM 1. Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_l\}$ be sets of non-zero integers. Let $\{p_1, \dots, p_s\}$ be a set of prime numbers. Suppose a_1, \dots, a_k are all composed of p_1, \dots, p_s . Let V be the set of ordered pairs (i, j) such that $a_i + b_j$ is composed of p_1, \dots, p_s if $(i, j) \in V$. Then

$$(1) \quad s \geq \frac{1}{4} \log \frac{|V|}{l} - 2.$$

Proof. Choose $j \in \{1, \dots, l\}$ such that there are at least $|V|/l$ integers i with $(i, j) \in V$. Taking $v = b_j$ we find that the equation $x + y = v$ has at least $|V|/l$ solutions $x = a_i + b_j, y = -a_i$. By applying Lemma 1 we obtain

$$|V|/l \leq 3 \times 7^{2s+3} < 7^{2s+4}$$

whence (1). ■

5. We now drop the condition that a_1, \dots, a_k are composed of p_1, \dots, p_s and derive a weaker lower bound for s .

THEOREM 2. Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_l\}$ be sets of integers. Let $\{p_1, \dots, p_s\}$ be a set of prime numbers. Let V be the set of ordered pairs (i, j) such that $a_i + b_j$ is composed of p_1, \dots, p_s if $(i, j) \in V$. If $|V| > k$ then

$$s \geq \frac{1}{4} \log \frac{|V|(|V| - k)}{kl^2} - 2.$$

We note that the assumption $|V| > k$ implies $l \geq 2$. It follows immediately from Theorem 2 that

$$\omega\left(\prod_{i=1}^k \prod_{j=1}^l (a_i + b_j)\right) \geq \frac{1}{4} \log(k/2) - 2$$

which is an explicit version of Theorem 1 of [5].

Proof. Let v_i be the number of integers j such that $(i, j) \in V$, for $i = 1, \dots, k$. Then the number of unordered pairs $(i, j_1), (i, j_2)$ in V^2 is $\sum_{i=1}^k \binom{v_i}{2}$.

By the Cauchy-Schwarz inequality we have

$$\begin{aligned} \sum_{i=1}^k \binom{v_i}{2} &= \frac{1}{2} \sum_{i=1}^k \left(v_i - \frac{1}{2}\right)^2 - \frac{k}{8} \geq \frac{1}{2k} \left(\sum_{i=1}^k \left(v_i - \frac{1}{2}\right)\right)^2 - \frac{k}{8} \\ &= \frac{1}{2k} \left(|V| - \frac{k}{2}\right)^2 - \frac{k}{8} = \frac{|V|(|V| - k)}{2k}. \end{aligned}$$

Since the number of unordered pairs j_1, j_2 is $\binom{l}{2}$, there exist two integers j and j' such that $(i, j) \in V$ and $(i, j') \in V$ for at least $|V|(|V|-k)/kl^2$ integers i . Taking $v = b_j - b_{j'}$, we find that the equation $x + y = v$ has at least $|V|(|V|-k)/kl^2$ solutions $x = a_i + b_j$, $y = -(a_i + b_{j'})$, each composed of p_1, \dots, p_s . By applying Lemma 1 we obtain that

$$\frac{|V|(|V|-k)}{kl^2} \leq 3 \times 7^{2s+3}. \quad \blacksquare$$

6. The estimates in Theorems 1 and 2 can be considerably improved when we assume that the numbers b_1, b_2, \dots, b_l are very large. Then it is possible to apply the following result which has been proved by means of estimates for linear forms in the logarithms of algebraic numbers. For a finite set $S = \{p_1, \dots, p_s\}$ of prime numbers and for a non-zero integer a , we shall denote by $[a]_{S^c}$ the largest divisor of a which is not divisible by p_1, p_2, \dots or p_s .

LEMMA 2. Let $S = \{p_1, \dots, p_s\}$ be a set of prime numbers not exceeding P and let v be a non-zero integer for which

$$[v]_{S^c} > \exp \{((2s)^s P)^c\}$$

for some appropriate effectively computable constant c . Then the equation $x + y = v$ has at most $4s + 1$ solutions in integers x, y composed of p_1, \dots, p_s .

This lemma is a special case of the Theorem of Győry [3].

Following the proof of Theorem 2, but using Lemma 2 instead of Lemma 1, we arrive at the following result.

THEOREM 3. Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_l\}$ be sets of integers where $l \geq 2$. Let $S = \{p_1, \dots, p_s\}$ be a set of primes not exceeding P . Suppose

$$[b_i - b_j]_{S^c} > \exp \{((2s)^s P)^c\}$$

for all pairs i, j with $1 \leq i < j \leq l$, where c is the constant occurring in Lemma 2. Let V be the set of ordered pairs (i, j) such that $a_i + b_j$ is composed of p_1, \dots, p_s if $(i, j) \in V$. Then

$$s \geq \frac{1}{4} \frac{|V|(|V|-k)}{kl^2} - \frac{1}{4}.$$

The essence of our method is to show that the sets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_l\}$ have subsets $A' \subseteq A$ and $B' \subseteq B$ such that $|B'| = 2$, $|A'|$ is "large" with respect to $|V|$ and for each $a \in A'$, $b \in B'$ the number $a + b$ is composed of p_1, \dots, p_s . Then Lemmas 1 and 2 can be applied. M. Simonovits has pointed out to us that a similar method has been used by Kővári, Sós and Turán [6] to solve a problem of Zarankiewicz.

7. We return to the graph theoretical problem posed by Erdős. Let $G(k)$ be a graph and a_1, \dots, a_k integers as described in the introduction such that $\omega(G(k))$ attains its minimal value. Let $\{p_1, \dots, p_s\}$ be the minimal set of prime numbers such that $a_i + a_j$ is composed of p_1, \dots, p_s if x_i and x_j are joined by an edge. Then $s = \omega(G(k))$. Choosing $l = k$ and $b_i = a_i$ for $i = 1, \dots, k$, we obtain the following consequence of Theorem 2.

COROLLARY 1. Let N be the number of edges of $G(k)$. If $N > k$ then

$$\omega(G(k)) \geq \frac{1}{4} \log \frac{N(N-k)}{k^3} - 2.$$

Put

$$\omega_-(G(k)) = \min \omega(\prod (a_i - a_j))$$

where the factor $a_i - a_j$ occurs if x_i and x_j are joined by an edge and the minimum is extended over all choices of distinct integers a_1, \dots, a_k . Choosing a_1, \dots, a_k such that $\omega_-(G(k))$ attains its minimal value and applying Theorem 2 with $l = k$ and $b_i = -a_i$ for $i = 1, \dots, k$, we obtain the following result.

COROLLARY 2. Let N be the number of edges of $G(k)$. If $N > k$ then

$$\omega_-(G(k)) \geq \frac{1}{4} \log \frac{N(N-k)}{k^3} - 2.$$

Corollaries 1 and 2 imply that if $N > k^{(3/2)+\varepsilon}$ then

$$\omega(G(k)) \geq_\varepsilon \log k \quad \text{and} \quad \omega_-(G(k)) \geq_\varepsilon \log k.$$

We conjecture that for any positive number ε the inequality $N > k^{1+\varepsilon}$ implies

$$\omega(G(k)) \geq_\varepsilon \log k \quad \text{and} \quad \omega_-(G(k)) \geq_\varepsilon \log k.$$

In our proofs the properties $\omega(G(k)) \rightarrow \infty$ and $\omega_-(G(k)) \rightarrow \infty$ as $k \rightarrow \infty$ are derived by showing that $G(k)$ contains a complete bipartite subgraph of the type $(u, 2)$ with u large. A graph is called a *complete bipartite graph of the type (u, w)* if its vertex set is the disjoint union of two distinct vertex sets U, W with $|U| = u, |W| = w$ such that two vertices are adjacent if and only if one of them belongs to U and the other to W .

COROLLARY 3. Let $G(k)$ be a graph as described in the introduction and suppose that $G(k)$ contains a complete bipartite subgraph of the type $(r, 2)$ with $r \geq 2$. Then

$$(2) \quad \min(\omega(G(k)), \omega_-(G(k))) \geq \frac{1}{4} \log(r/2) - 2.$$

The lower estimate for $\omega_-(G(k))$ is a considerable improvement of the special case $K = \mathbf{Q}, N = 1$ of Theorem 2 of Győry [4] which concerns certain closely related arithmetical graphs. We intend to return to these arithmetical graphs in a later paper.

Proof of Corollary 3. Suppose that $\{X_{i_1}, \dots, X_{i_r}\}, \{X_{j_1}, X_{j_2}\}$ are the disjoint vertex sets of a complete bipartite subgraph of the type $(r, 2)$ of $G(k)$. Let a_1, \dots, a_k be distinct integers. By applying Theorem 2 first to the sets $\{a_{i_1}, \dots, a_{i_r}\}, \{a_{j_1}, a_{j_2}\}$ and then to the sets $\{a_{i_1}, \dots, a_{i_r}\}, \{-a_{j_1}, -a_{j_2}\}$, (2) immediately follows in both cases. ■

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