# Algebra comprehensive exam

January 23, 2018

Answer all questions, show all your work, and justify any statements that you make.

1. Consider the $3 \times 3$ matrix with entries in $\mathbb{Q}$
$$A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix}$$
   (a) Describe a field extension $F$ of $\mathbb{Q}$ of minimal degree (either abstractly, or as a subfield of the complex numbers), such that $A$ has an eigenvector with entries in $F$ (note: you do **not** need to find the eigenvector or eigenvalue).
   (b) Determine if $A$ is diagonalizable over $\mathbb{C}$.
   (c) Does there exist a $3 \times 3$ matrix with rational coefficients with no eigenvectors over $\mathbb{Q}$ which is not diagonalizable over $\mathbb{C}$? Find an example of such a matrix, or prove none exists.
2. Let $V$ be an $n$-dimensional vector space over a field $F$ and let $A\colon V \to V$ be a linear transformation whose minimal polynomial $m_A$ is of degree 2. Consider $V$ as a module over $F[x]$ where $x$ acts by $A$.
   (a) List the possible isomorphism types of $V$, for each possible factorization of $m_A$ into irreducibles.
   (b) Show that if $m_A$ has a root, then there is an eigenvalue $\lambda$ such that the eigenspace has dimension $\geq n/2$.
3. Let $f(x) = x^4 - 3$.
   (a) Describe a splitting field $E$ for $f(x)$ over $\mathbb{Q}$ as $\mathbb{Q}(a_1, \dots)$ for $a_i \in \mathbb{C}$.
   (b) Determine the Galois group $\mathrm{Aut}(E/\mathbb{Q})$ and how it acts on the generating elements you've given.
   (c) Is this group a symmetric group or dihedral group? Prove your answer.
4. Let $K/L/F$ be a tower of fields, such that $K = F(\alpha)$ for some element $\alpha \in K$. Let $m(x) = x^n + a_1x^{n-1} + \cdots + a_n$ be the minimal polynomial of $\alpha$ over $L$. Show that $L = F(a_1, \dots, a_n)$.
5. (a) Let $G$ be a group of order $mp$ where $m$ and $p$ are coprime. Show that if $G$ has $k$ $p$-Sylow subgroups, then $G$ has precisely $k(p-1)$ elements of order $p$.
   (b) Assume that $P$ is a normal $p$-Sylow subgroup of $G$. Show that if $H$ is a subgroup of $G$ of order coprime to $p$, then $HP$ is a subgroup isomorphic to a semi-direct product $H \ltimes P$.
   (c) Classify groups of order 30 up to isomorphism.
6. Let $G$ be a group, and $H$ a subgroup of finite index $n$. Prove or give a counterexample to the following statements:
   (a) If $a \in G$, then $a^n \in H$.
   (b) If $a \in G$, then for some $0 < k \leq n$, we have $a^k \in H$.
7. (a) Give a complete and irredundant list of abelian groups of order 144.
   (b) Give a complete and irredundant list of finitely generated modules over $\mathbb{F}_2[t]$ where the polynomial $t^4 + t^3 + t + 1$ acts trivially.
8. We call a ring Artinian if it satisfies the *descending chain condition*, that is, there is no infinite descending sequence of ideals $I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$.
   (a) Show that any commutative Artinian domain is a field.
   (b) Show that if $R$ is a PID, and $R \to S$ is a surjective ring homomorphism, then either $R \cong S$ or $S$ is Artinian.