Pure Mathematics Algebra Comprehensive Examination
University of Waterloo
Term: Winter Year: 2020

Instructions

1. Print your name and UWaterloo ID number at the top of this page.
2. Check for questions on both sides of each page.
3. There are eight questions that are equally weighted. Please answer all eight questions.
4. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
5. Do not write on the Crowdmark QR code at the top of each page.
6. Use a dark pencil or pen for your work.

Linear algebra

1. Let V be a finite-dimensional vector space over a field F of characteristic $\neq 2$. Let $X, Y : V \to V$ be linear transformations such that $X^2 = Y^2 = \mathrm{Id}$, where $\mathrm{Id} : V \to V$ is the identity transformation.

   (a) Show that X and Y are diagonalizable, with eigenvalues in $\{\pm 1\}$.

   (b) Show that if $YX = \lambda XY$ for some $\lambda \in F$, then $\lambda \in \{\pm 1\}$.

   (c) Suppose that $XY = -YX$. Show that $\dim V$ is even, and that there is a basis of V in which the matrices of X and Y are

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \text{ and } \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

   respectively, where I_n is the $n \times n$ identity matrix and $\dim V = 2n$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Linear algebra

2. Given finite-dimensional vector spaces V and W over a field F, show that there is an isomorphism $\phi : \mathrm{Lin}(V, W) \to W \otimes_F V^\vee$ such that

$$\phi(STR) = (S \otimes R^\vee)\ \phi(T)$$

for all $S \in \mathrm{Lin}(W, W)$, $T \in \mathrm{Lin}(V, W)$, and $R \in \mathrm{Lin}(V, V)$.

(Here $\mathrm{Lin}(V, W)$ denotes the space of linear maps from V to W, $V^\vee$ denotes the dual space $\mathrm{Lin}(V, F)$, and $R^\vee \in \mathrm{Lin}(V^\vee, V^\vee)$ denotes the dual operator to R.)

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Group theory

3. Prove or find a counterexample:

   (a) Every group of order 15 is abelian.

   (b) Every group of order 8 is abelian.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Group theory

4. Let

$$G = \langle x, y, z : xy = zyx, xz = zx, yz = zy \rangle,$$

and let

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$$

be the group of invertible upper-triangular 3×3 integer matrices with 1's on the diagonal.

(a) Show that every element of G can be written in the form $z^c y^b x^a$ for some $a, b, c \in \mathbb{Z}$.

(b) Show that there is an isomorphism $\phi : G \to H$ with

$$\phi(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \phi(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \phi(z) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Find the commutator subgroup $G^{(1)}$ of G, and use this to show that H is nilpotent and that the abelianization $H/H^{(1)}$ of H is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Ring Theory

5. In each of the following cases, give, with justification, an example of a ring having the stated properties.
   (a) A commutative ring that is not an integral domain.
   (b) An integral domain that is not a unique factorization domain (UFD).
   (c) A UFD that is not a principal ideal domain (PID).
   (d) A nontrivial PID that is not isomorphic to $\mathbb{Z}$, nor to a field, nor to a polynomial ring over a field.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Ring Theory

6. Suppose $I_1, \ldots, I_n$ are ideals of a commutative ring A such that

$$I_1 \cap \cdots \cap I_n = (0).$$

Prove that if each A/I_i is a noetherian ring then A is a noetherian ring.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Field Theory

7. What are the possible Galois groups of a cubic polynomial $f \in \mathbb{Q}[x]$? Justify your answer, and for each possible Galois group G give an example of a polynomial f whose Galois group is G.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Field Theory

8. Suppose F is a field and t is an indeterminate. In the rational function field $F(t)$, let $s := \dfrac{t^3}{t+1}$.

   (a) Prove that s is transcendental over F.

   (b) Prove that $F(t)$ is a simple algebraic extension of $F(s)$.

   (c) Compute $[F(t) : F(s)]$.

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