Algebra Comprehensive Examination

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Do at least six questions. All have approximately equal value.

1. Prove or disprove:

a) $\mathbb{R} \cong \mathbb{R}^2$ as vector spaces over $\mathbb{R}$.

b) $\mathbb{R} \cong \mathbb{R}^2$ as vector spaces over $\mathbb{Q}$.

c) $\mathbb{R} \cong \mathbb{R}^2$ as abelian groups.

d) If G is a finite group whose order is divisible by the prime power p^m, then G has an element of order p^m.

e) Every subgroup of a finitely generated group is also finitely generated.

f) If $G_1 \times G_2 \cong H_1 \times H_2$, then either $G_1 \cong H_1$ or $G_1 \cong H_2$, for finite groups G_i, H_i.

g) If F is an ordered field, then x^2+1 is irreducible in $F[x]$.

2. a) Let F be any field. If A and B are in the ring $M_n(F)$ of $n \times n$ matrices over F, when are A and B said to be similar? What is the minimal polynomial of A?

b) Give an example of A and B as above which are <u>not</u> similar, yet have the same eigenvalues occurring with the same multiplicities.

c) Prove that no example as in b) could consist of two symmetric matrices when $F = \mathbb{R}$. State clearly any theorem(s) used.

d) What set of invariants of $A \in M_n(F)$ will completely determine its similarity class?

3. What is a <u>composition series</u> for a finite group? To what extent is such a series unique? Write down such a series for S_n, the symmetric group, when $n \geq 5$. Explain briefly the significance of this last series for the solvability by radicals of polynomial equations.

4. a) Find all invertible elements in the ring $\mathbb{Z}[\sqrt{-5}]$.

   b) Give an example of a unique factorization domain which is not a principal ideal domain.

   c) Define the term <u>maximal left ideal in a ring</u> and prove that every non-zero ring unity has one.

   d) Show that the intersection of all maximal left ideals in R is a two sided ideal of R.

   e) Find a non-zero maximal left ideal in $M_n(F)$, but show that the intersection above is the zero ideal in this case.

5. a) Show that if the fields $F \subset K \subset L$ are such that K is algebraic over F and L is algebraic over K, then L is algebraic over F.

   b) What is the cardinality of the algebraic closure of $\mathbb{Q}$?

   c) Determine $[K:\mathbb{Q}]$ where K is the splitting field of x^6-8 over $\mathbb{Q}$.

   d) What is the Galois group of $f(x) \in \mathbb{Q}[x]$? Prove: When $f(x) = x^n-1$ the Galois group is isomorphic to a subgroup of the group of invertibles in $\mathbb{Z}/n\mathbb{Z}$.

   e) For each positive integer k, how many fields are there (up to isomorphism) of order k?

6. Let M', M, M'' be modules over a commutative ring R with unity. Prove that if

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is an exact sequence then for any R-module N

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \longrightarrow 0$$

is exact. Give an example to show that it is not true in general that

6. (cont'd)

if

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is an exact sequence of R-modules then

$$0 \longrightarrow M' \otimes N \xrightarrow{f\otimes 1} M \otimes N \xrightarrow{g\otimes 1} M'' \otimes N \longrightarrow 0 \qquad (*)$$

is an exact sequence for any R-module N.

Show that an R-module is projective if and only if it is the the direct summand of a free R-module. Give an example of a projective module which is not free. Show that if N is projective then (*) is exact.

7. a) State the structure theorem for finitely generated abelian groups.
   b) Prove that a finite multiplicative subgroup of non-zero elements in a field is cyclic.
   c) Show that the group of invertible elements of the ring $\mathbb{Z}/p^m\mathbb{Z}$ is cyclic, if p is an odd prime.
   c) Let A be a <u>proper</u> subgroup of a free abelian group G. Prove that there is a homomorphism $\theta: G \to \mathbb{R}$ such that $\theta(A) \subset \mathbb{Z}$ but $\theta(G) \not\subset \mathbb{Z}$.
   (Hint: Any non-zero abelian group admits a non-zero homomorphism into the circle group).

8. Show that every subgroup and every factor group of a nilpotent group is nilpotent. Give an example of a group G with normal subgroup N such that both G/N and N are nilpotent but G is not nilpotent. Show that every nilpotent group is solvable. Give a counterexample

8. (cont'd)

to the converse of the last statement.

9. a) Define the exterior algebra $\Lambda(V)$ of a vector space V.

   b) If V has finite dimension n, what is the dimension of $\Lambda(V)$ as a vector space?

   c) F ind a maximal two sided ideal in $\Lambda(V)$.

   d) If $T: V \to V$ is a linear endomorphism, what is the matrix of $\Lambda^n(T): \Lambda^n(V) \to \Lambda^n(V)$ with respect to some basis, where $n = \dim(V)$?