PURE MATHEMATICS DEPARTMENT

UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

# ALGEBRA COMPREHENSIVE EXAMINATION, SPRING 1988

TIME: 3 HOURS

ANSWER 6 QUESTIONS. ANSWER AT LEAST ONE FROM EACH OF THE FOLLOWING GROUPS:

$\{1,2,3\}$, $\{4,5\}$, $\{6,7,8\}$, $\{9,10,11\}$.

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1.(a) Let $G$ be a group and $Z(G)$ the centre of $G$. If $G/Z(G)$ is cyclic, prove that $G$ is abelian.

(b) If $G$ is non-abelian of order $pq$ where $p$ and $q$ are distinct primes, prove that the centre of $G$ is trivial.

(c) If $G$ is non-abelian, show that $\mathrm{Aut}(G)$ is not cyclic.

(d) For every integer $n$ and every prime $p$, show that $n^p \equiv n \pmod p$.

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2.(a) Define "normal subgroup of a group".

(b) If $H \triangleleft K$, $K < G$, is it true that $H \triangleleft G$? Explain.

(c) Define "solvable group". If $G$ is solvable and $H \triangleleft G$, show that $G/H$ is solvable.

(d) For $n \geq 5$, prove that $S_n$ is not solvable. (Hint: $A_n$ is simple for $n \geq 5$).

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3.(a) Let $G$ be a finite group and let $C(a)$ be the centralizer of $a$ in $G$. Show that $|G| = \Sigma[G: C(a)]$, where the sum runs over one element from each conjugacy class of $G$.

(b) Show that a group of order $p^2$ is abelian.

(c) If a finite group $G$ is abelian and a prime $p$ divides the order of $G$, then show (without assuming the structure theorem) that $G$ has an element of order $p$.

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4.(a) Show that the eigenvalues of a hermitian matrix are real.

(b) Find a nonsingular matrix $P$ such that $P^*AP$ is diagonal where

$$A = \begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{pmatrix},$$

($P^*$ is the conjugate transpose of $P$.)

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5.(a) Let $T: V \to V$ be linear. Suppose, for $v \in V$, $T^k(v) = 0$ but $T^{k-1}(v) \neq 0$. Prove that i) The set $S = \{v, T(v), \ldots, T^{k-1}(v)\}$ is linearly independent. ii) The subspace $W$ generated by $S$ is T-invariant.

(b) Let $A = \begin{pmatrix} -1 & -2 & -1 \\ -1 & -1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$. Determine the Jordan canonical form $J$ of $A$ and a matrix $C$ such that $C^{-1}AC = j$.

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6. Describe the field of constructible numbers in $\mathbb{C}$ (constructible by straight-edge and compass) and show that the regular pentagon is constructible.

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7. Find the Galois group of $x^5-5x-1$ (over $\mathbb{Q}$) and determine whether or not the roots can be found by radicals.

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8.(a) If $F_1$ and $F_2$ are two countable algebraically closed fields of characteristic $0$, show that $F_1 \longrightarrow F_2$ or $F_2 \longrightarrow F_1$.

(b) How many isomorphism types of countable algebraically closed fields of characteristic $0$ exist?

(c) If $F_1$ and $F_2$ are uncountable algebraically closed fields of characteristic $0$ and $|F_1| = |F_2|$, prove that $F_1 \cong F_2$.

(d) Prove that $|\mathrm{Aut}(\mathbb{C})| = 2^{|\mathbb{C}|}$.

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<u>Note.</u> In the following questions, "ring" means "ring with 1".

9.(a) Let $I_1,\ldots,I_n$ be ideals of a ring $R$ such that $I_i + I_j = R$ for $i \neq j$. Show that

$$R/\bigcap_{j=1}^{n} I_j \cong \prod_{j=1}^{n} (R/I_j)$$

(b) Give a number-theoretic interpretation of the above theorem in the case $R = \mathbb{Z}$.

10. Which of the following commutative rings are unique factorization domains? Justify your answers.

$\mathbb{Z}[X,Y]$, $\mathbb{Z}[\sqrt{-3}]$, $\mathbb{C}[X,Y]/(X^2+Y^2-2)$, $\mathbb{Z}[X,Y]/(X^2+Y^2,2)$, $\mathbb{Q}[X,X^{-1}]$.

11. Prove that the intersection of the maximal left ideals of a ring $R$ is a two-sided ideal of $R$.