UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

The Comprehensive Examination in Algebra

Department of Pure Mathematics

May 5, 1982

TIME: 3 HOURS

Answer all questions.

GENERAL

1. a) Identify all abelian groups of order 360 up to isomorphism.

   b) Let G be the cyclic group of order 6 and let H be the cyclic group of order 18. List the homomorphisms from G to H.

   c) List the homomorphisms of rings with one from Z_6 to Z_{18}.

   d) Compute $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{3721}$.

   e) How many nonisomorphic fields are there of order

      (i) 81?

      (ii) 36?

   f) Suppose A and B are countably infinite abelian groups, each satisfying the additional equation $5x = 0$. Prove that A is isomorphic to B.

...con'd/2

2. (a) Define "Sylow p-subgroup" of a group G. Then state the basic results on Sylow p-subgroups.

   (b) Prove that any group G of order pq, where p and q are distinct primes with $p \not\equiv 1 \pmod q$ and $q \not\equiv 1 \pmod p$, is cyclic.

   (c) Prove that no group of order 56 is simple.

3. (a) Prove that a Euclidean domain is a principal ideal domain.

   (b) Prove that $Z[\sqrt{2}]$ is a Euclidean domain with respect to the norm $N(m+n\sqrt{2}) = |m^2-2n^2|$.

   (c) Outline a proof that a principal ideal domain is a unique factorization domain giving the main ideas only.

4. Let V be a vector space over a field F, $T: V \to V$ a linear transformation and let $F[X]$ be the polynomial ring over F in an indeterminate X.

   (a) Show that V becomes an $F[X]$-module if we define

$$(a_0+a_1X+\dots+a_nX^n)v = a_0v+a_1Tv+\dots+a_nT^nv.$$

   (b) Let V have a countable basis $\{w_1,w_2,\dots\}$ and let T be the linear transformation defined by $Tw_1 = 0$, $Tw_{i+1} = w_i$, $i = 1,2,\dots,n,\dots$. Show that V as an $F[X]$-module defined by T as in (a) is Artinian.

   (c) Let T' be the linear transformation from V to V such that $T'w_i = w_{i+1}$, $i = 1,2,\dots$. Show that V as an $F[X]$-module from T' is isomorphic to $F[X]$. Is V noetherian?

...con'd/3

5. Determine the splitting field E of x^3-2 over Q and calculate $[E:Q]$. Find the Galois group G of x^3-2, and for each subgroup G_i of G describe the corresponding subfield F_i of E as an extension of Q by suitable elements.

6. Determine the rational and Jordan canonical forms for

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{bmatrix}$$

in $M_4(Q)$.