final version.

Ph.D. Comprehensive Examination October 21, 1988

ALGEBRA

Note: All rings are associative with 1. Subrings inherit the same 1.

1. a) Define: prime ideal, maximal ideal.

   b) Prove that in a commutative ring R, M is a maximal ideal if and only if R/M is a field.

   c) State Hilbert's Nullstellensatz (which characterizes the maximal ideals in a polynomial ring $F[x_1,\ldots,x_n]$ over an algebraically closed field F).

   d) Let P be a non-principal prime ideal of the polynomial ring $\mathbb{C}[x,y]$. Prove that P is maximal.

2. a) Describe (without proof) the Jacobson radical of each of the following rings:

   i) $M_2(\mathbb{C})$

   ii) $R = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) : \quad c = 0\right\}$ (upper triangular matrices)

   iii) $R[t]$ (R above, t a commuting indeterminate)

   iv) $\mathbb{C}[[t]]$ (power series ring).

   b) Prove that if the matrix rings $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$ are isomorphic (as rings) then $n = k$.

   c) Prove that if the ring $M_n(\mathbb{C})$ is isomorphic to a subring of $M_k(\mathbb{C})$, then n divides k. (Hint: Use the matrix rank.)

3. a) Define: solvable group and solvable length.

   b) Use the Sylow theorems to prove that every group of order 12 is solvable.

   c) Let $\alpha,\beta\colon \mathbb{Z} \to \mathbb{Z}$ be functions defined by $\alpha(x) = x+1$, $\beta(x) = -x$. Prove that the group generated by these two functions is solvable.

4. a) Define: divisible (Abelian) group.

   b) How many isomorphism classes of countable torsion-free divisible groups are there. Justify your answer.

   c) Prove (from basic principles) that there are no non-trivial finitely generated divisible groups.

5. (a) Define the field F of (ruler and compass) constructible complex numbers.

   (b) What are the possible degrees of the minimum polynomial $p(x) \in Q[x]$ of an element of F?

   (c) State and prove Eisenstein's irreducibility test.

   (d) For p a prime number show that $e^{2\pi i/p} \in F$ implies p is a Fermat prime (i.e., of the form $2^{2^n}+1$). [Equivalently, show that if one can construct the regular p-gon then p is a Fermat prime.]

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6. (a) Describe the splitting field E of x^3-2 over Q.

   (b) Calculate $[E:Q]$.

   (c) Find the Galois group G of x^3-2.

   (d) For each subgroup G_i of G describe the corresponding subfield F_i of E as an extension of Q by suitable elements.

7. Determine the rational and Jordan canonical forms for

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{bmatrix} \quad \text{in } M_4(Q).$$

8. Let T be a normal operator on a complex inner product (unitary) space V, and let $u, v \in V$, $c_i \in \mathbb{C}$. Show

(a) $|Tu| = |T^*u|$

(b) $Tu = cu \Rightarrow T^*u = \bar{c}u$

(c) $Tu = c_1u$, $Tv = c_2v$, $c_1 \neq c_2 \Rightarrow (u,v) = 0$

(d) $(T^m)u = 0$ for some $m > 1 \Rightarrow Tu = 0$

(e) $g(x) \in \mathbb{C}[x] \Rightarrow g(T)$ is normal.