UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

The Comprehensive Examination in Algebra

Department of Pure Mathematics

April 30, 1981

Time: 3 Hours

Answer as many as possible.

All rings are rings with 1.

MARKS

6 1. If the nonzero integers $n_1, n_2, \ldots, n_k$ are pairwise coprime and $n = n_1 n_2 \ldots n_k$, show that we have a ring isomorphism

$$Z/(n) \cong Z/(n_1) \times \ldots \times Z/(n_k).$$

2. (a) Give the definitions of the following:
   - an (commutative) integral domain,
   - an irreducible element of an integral domain,
   - a prime element of an integral domain,
   - a principal ideal domain (PID)
   - a unique factorization domain (UFD).

   (b) Give an example of a UFD which is not a PID. (Give reasons.)

10 3. (a) Let H be a normal subgroup of G. If H and G/H are solvable show that G is solvable.

   (b) If G is a group of order pq, where p and q are primes, show that G is solvable. (You may use the Sylow theorems.)

RKS

6 4. (a) Let G be a group and H a subgroup of index 2. If $x,y \in G$ are not in H show that $xy \in H$.

(b) Use part (a) to show that the alternating group A_4 has no subgroup of order 6.

(c) If G is a subgroup of S_n (the symmetric group) containing an odd permutation show that G has a subgroup of index 2.

10 5. Let G be a finite abelian group of order n.

(a) State the structure theorem for G in the case when n is a power of a prime.

(b) Using part (a) and the Sylow theorem show that, in the general case, there exists a direct decomposition

$$G = G_1 \times G_2 \times \ldots \times G_m$$

such that G_i is cyclic of order h_i $(1 \le i \le m)$, $h_1 > 1$, and h_i divides h_{i+1} for $1 \le i \le m-1$.

(c) If m is the integer from part (b), show that every generating set of G has at least m elements.

8 6. Let G be a finite non-cyclic abelian group. Show that $\mathrm{Aut}(G)$ is not abelian.

10 7. Prove that the multiplicative group Z_p^* of the non-zero elements of the field $Z_p = Z/(p)$, p a prime, is cyclic.

MARKS

8 8. Let R be a non-zero commutative ring and F a finitely generated free R-module. Show that:

(a) Every basis of F is finite.

(b) Any two bases of F have the same number of elements.

8 9. Let R be a non-zero ring such that every (left) R-module is free. Show that R is a division ring.

10 10. Let K be the splitting field of $X^3 - 7$ over Q. Obtain the lattice diagram for the subfields of K and indicate the corresponding subgroups of the Galois group G of K/Q.

12 11. Let A be a complex n by n matrix.

(a) Show that there exists a unitary matrix U such that UAU^* is upper triangular.

(b) If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and if $f(t)$ is a polynomial show that the eigenvalues of the matrix $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

10 12. Let V and W be complex vector spaces, $\dim V = m$, $\dim W = n$, and let $T_1 : V \to V$ and $T_2 : W \to W$ be linear transformations.

(a) Show that there is a unique linear transformation $T: V \otimes W \to V \otimes W$ such that $T(x \otimes y) = T_1(x) \otimes T_2(y)$ for all $x \in V$ and $y \in W$.

(b) If $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ are the eigenvalues of T_1 and T_2, respectively, determine the eigenvalues of T.