UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

ALGEBRA COMPREHENSIVE EXAMINATION

PURE MATHEMATICS DEPARTMENT

TIME: 3 HOURS MAY 20, 1986

ANSWER AS MANY QUESTIONS AS POSSIBLE. ANSWER AT LEAST ONE QUESTION FROM EACH OF THE SETS {1,2,3}, {4,5,6}, {7,8,9}, {10,11,12}.

1.(a) Let A and B be $n\times n$ complex matrices with the same characteristic polynomial $(x-c_1)^{n_1}\cdots(x-c_k)^{n_k}$ (where $c_1,\ldots,c_k$ are distinct) and the same minimal polynomial. Show that if every $n_i \le 3$ then A and B are similar.

LA

Give an example to show that this result is false if 3 is replaced by 4.

(b) Let α be an algebraic number of degree n and let the operator T on $\mathbb{Q}(\alpha)$ (where $\mathbb{Q}(\alpha)$ is considered as a vector space over $\mathbb{Q}$) be defined by $T(x) = \alpha x$. Let A be the matrix of T relative to some basis of $\mathbb{Q}(\alpha)$. Describe the Jordan canonical form of A (where A is considered as a complex matrix).

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2. V is a finite-dimensional complex inner product space. Define "normal operator" and "orthogonal projection" for V.

LA

State the spectral theorem for normal operators on V and discuss whether its converse is true.

Prove that an operator T on V is normal iff $\|T(x)\| = \|T^*(x)\|$ for all x in V.

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3. K is a field and U,V are vector spaces over K.

(i) Define the tensor product $U \otimes V$ of U and V.

MLA

(ii) Prove that $K \otimes V \cong V$.

(iii) Show how linear transformations $S: U \to U'$ and $T: V \to V'$ induce a linear transformation $S \otimes T: U \otimes V \to U' \otimes V'$.

3.(cont'd)

(iv) Prove that if $\sum_{i=1}^{n} x_i \otimes y_i = 0$ in $U \otimes V$ then either every $x_i = 0$, or $y_1, \ldots, y_n$ are linearly dependent.

(v) Prove that if U and V are finite-dimensional then $\dim(U \otimes V) = \dim U \cdot \dim V$.

4.(a) E is an algebraic extension of a field K and R is a subring of E containing K. Prove that R is a field.

(b) E is as in (a), M and N are subfields of E containing K of finite degrees m and n respectively over K, and MN denotes the smallest subfield of E containing both M and N. Prove that MN is of finite degree $\leq mn$ over K.

FT

Give an example to show that inequality may occur here even if $M \cap N = K$.

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5. Let $f(x)$ be a irreducible polynomial of degree 4 over $\mathbb{Q}$, let G be the Galois group of $f(x)$ over $\mathbb{Q}$, and let α be a root of $f(x)$. Show that there is no field lying strictly between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$ iff $G \cong A_4$ or $G \cong S_4$.

FT

6.(a) Define "algebraically closed field" and "the algebraic closure of a field". What is the relationship between the cardinality of a field K and the cardinality of its algebraic closure?

FT

(b) Prove that the only automorphism of the real field $\mathbb{R}$ is the identity automorphism.

Discuss the number of automorphisms of the complex field $\mathbb{C}$.

7.(a) Prove that if G is a finitely generated group then every proper subgroup of G is contained in a maximal (proper) subgroup of G.

(b) Is the above true if we remove the condition that G be finitely generated? Explain.

GT

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8. Prove that no group of order 88 is simple. Is such a group necessarily abelian? Justify your answer.

GT

9.(a) Define "divisible abelian group" and state the structure theorem for divisible abelian groups.

GT

(b) If A and B are divisible abelian groups and each is isomorphic to a subgroup of the other, prove that A and B are isomorphic.

(c) By considering the restricted Cartesian product G of $\aleph_0$ copies of $\mathbb{Z}_4$, and $\mathbb{Z}_2 \times G$, show that the statement in (b) does not hold for abelian groups in general.

10.(a) State the Hilbert Basis Theorem (in some reasonable form).

(b) Prove that in commutative Noetherian integral domain, each nonzero nonunit element is a product of irreducible elements (an element of an integral domain is <u>irreducible</u> if it is nonzero, nonunit, and is not the product of two nonunits).

RT

(c) Prove that $R = \mathbb{C}[x,y]/\langle x^2+y^2-1\rangle$ is a Noetherian integral domain.

(d) Is R (as in (c)) a principal ideal domain? Is it a unique factorization domain? Justify your answers.

11.(a) Define the Jacobson radical and prime radical of a ring.

(b) Let R be the ring of upper triangular $n \times n$ matrices over $\mathbb{Z}$. Describe the Jacobson radical of R.

RT

(c) Show that if R is a commutative integral domain then the polynomial ring $R[t]$ has zero Jacobson radical.

11.(cont'd)

(d) Prove or disprove: "If the ring R has no zero-divisors then R has zero Jacobson radical".

12.(a) Define "Noetherian module" and "Artinian module".

(b) Describe all the Noetherian modules and Artinian modules over the ring R of 2×2 real matrices.

R T (c) Describe (with proof) all $\mathbb{Z}$-modules which are both Artinian and Noetherian.

(d) Prove that each finite subgroup of the multiplicative group of a field is cyclic.
