Algebra Comprehensive Examination

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Do six questions. All questions have equal value.

1.(a) Let N be the normalizer of a subgroup H of a group G. Show that the number of distinct conjugates of H in G is $[G:N]$.

(b) Let A,B be subgroups of G and $H = \langle A,B \rangle$ the subgroup of G generated by A and B. Prove that

$$[A: A \cap B] \leq [H:B].$$

2.(a) Define nilpotent and solvable groups.

(b) Let G be a finite nilpotent group. Show that G is the direct product of its Sylow subgroups.

[Hint: You can assume that the normlizer of sylow-p subgroup in G is its own normalizer in G.]

(c) Let G be finite of order n. Let $m|n$. If G is nilpotent show that G has a subgroup of order m. Is this true if G is solvable? Prove or disprove the statement.

3.(a) Prove that in a nonzero ring with] every ideal is contained in a maximal ideal.

(b) Let $F[[x]]$ denote the ring of formal power series in the indeterminate x over a field F. Prove that

(i) $a_0 + a_1x + a_2x^2 + \ldots \in F[[x]]$ has an inverse if $a_0 \neq 0$.

(ii) $F[[x]]$ has a unique maximal ideal.

4. Let R be a commutative ring with 1.

(a) Define prime element and irreducible element of R. Give an example of a ring in which a prime element is not irreducible and an example of a ring in which an irreducible element is not prime. (State your reason briefly.) If R is an integral domain, show that every prime element is irreducible.

(b) Let S be a subset closed under multiplication such that there exists a nonempty set C of ideals disjoint from S. If M is a maximum element in C show that M is a prime ideal.

5.(a) State the fundamental theorem of Galois theory.

(b) Find the Galois group of the polynomial $f(x) = x^3 - 3x - 3$ over $\mathbb{Q}$.

(c) Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Suppose a_1, a_2, a_3 are the roots of $f(x)$. Find all the normal extensions of $\mathbb{Q}$ contained in K.

6.(a) Let F be a field of char $p \neq 0$. Let $f(x) = x^p - a$ where $a \in F$ and $a \neq b^p$ for all $b \in F$. Show that $f(x)$ is irreducible in $F[x]$.

(b) Let $F = Z_p(\alpha)$ be the field extension of Z_p by the transcendental element α. Show that $f(x) = x^p - \alpha$ is irreducible and inseparable in $F[x]$.

7.(a) Prove that a symmetric operator on a real finite dimensional inner product space has a set of eigenvectors which form an orthonormal basis.

(b) Find the canonical form for orthogonal operators on a finite dimensional real inner product space.

8. Let V be a finite dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation

(a) Find all the linear transformations T with the property that every subspace of V is a T-invariant subspace.

(b) Let T be diagonizable with eigenvalues $\lambda_1,\ldots,\lambda_k$. Let $W_1,\ldots,W_k$ be the eigenspaces belonging to $\lambda_1,\ldots,\lambda_k$ resp. If U is a T-invariant subspace of v show that

$$U = U \cap W_1 \oplus \ldots \oplus U \cap W_k.$$