PURE MATHEMATICS DEPARTMENT

ALGEBRA COMPREHENSIVE EXAMINATION

MAY, 1989

TIME: 4 hours

EXAMINERS: D.Ž. Djoković, D.A. Higgs

INSTRUCTIONS: Answer 6 questions, at least one from each of the following groups: {1,2}, {3,4}, {5,6}, {7,8}.

<u>NOTE:</u> ALL RINGS ARE ASSOCIATIVE WITH 1.

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1. (a) Let G be a finite abelian group with the property that, for each positive integer n, the equation $x^n = 1$ has at most n solutions in G. Using either the structure theorem for finite abelian groups or a direct argument, prove that G is cyclic. What does this tell you about the multiplicative group of a field?

GT

   (b) Let A be a torsion abelian group (torsion = every element has finite order). Prove that A is the direct sum $\bigoplus_p A_p$ of its subgroups A_p, p prime, where A_p consists of the elements of A whose order is some power of p.

2. Let H be a subgroup of a finite group G.

   (a) Prove that $N(H) = \{x \in G: x^{-1}Hx = H\}$ is a subgroup of G which contains H as a normal subgroup.

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   (b) Prove that the number of distinct conjugates $x^{-1}Hx$ of H in G equals the index of $N(H)$ in G.

   (c) Prove that if $H \neq G$ then there exists an element a of G such that $xax^{-1} \notin H$ for all $x \in G$.

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3. (a) In the context of commutative rings, define "ideal", "prime ideal", "prime element", "unique factorization domain".

(b) Let R be an integral domain and let S be a multiplicatively closed subset of R. Prove that if I is an ideal of R which is disjoint from S and is maximal with respect to this property, then I is a prime ideal of R.

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(c) Let R be an integral domain in which every non-zero prime ideal contains a prime element. Use part (b) to prove that R is a unique factorization domain.

(d) Is it true that if R is a unique factorization domain then every prime ideal of R is principal? Justify your answer.

4. Let J denote the Jacobson radical of a ring R.

(a) Prove Nakayama's Lemma stating that if X is a finitely generated left R-module for which $JX = X$ then $X = \{0\}$.

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(b) Prove that if R is left Artinian then J is nilpotent.

5. (a) Let $E \subseteq L$ be a (finite) Galois field extension. Describe carefully, but without proofs, the Galois correspondence between the fields K such that $E \subseteq K \subseteq L$ and the subgroups H of the Galois group G of L over E.

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(b) Set up this Galois correspondence for the following instance: $E = \mathbb{Q}$, $L =$ the splitting field of x^4-2 over $\mathbb{Q}$.

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6. (a) Let $E \subseteq K \subseteq L$ be a tower of fields where K is algebraic over E and L is algebraic over K. Prove that L is algebraic over E.

(b) Define "transcendence basis" and "transcendence degree" for a field extension $E \subseteq L$.

(c) Let $E \subseteq K \subseteq L$ be a tower of fields where L is finitely generated over E. Prove that K is finitely generated over E.

7. (a) Let V be a finite-dimensional vector space over a field F and let T be a linear operator on V. Prove that T is triangulable iff the minimum polynomial of T is a product of linear polynomials over F. LP

(b) State and prove a similar necessary and sufficient condition for T to be diagonalizable.

8. (a) Let $f: M_n(F) \to F$ be a linear function such that $f(AB) = f(BA)$ for all $A,B \in M_n(F)$. ($M_n(F)$ is the space of $n \times n$ matrices over a field F.) Show that $f(A) = \lambda$ trace(A) for all matrices A and a suitable constant $\lambda \in F$. LP

(b) If A and B are complex $n \times n$ matrices and $ABB^* = 0$, show that $AB = 0$. (B^* = the conjugate transpose of B.)

(c) Show that the quadratic forms $2x_1^2+x_2^2+x_3^2+2x_1x_2+2x_1x_3+2x_1x_4$ and $y_1y_3+y_2y_4$ have the same rank. Can they be transformed into one another by a real linear transformation? Justify your answer.