UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

Algebra Comprehensive Examination

Pure Mathematics Department

Time: 3 Hours October 28, 1985

1-4 p.m.

Answer 9 questions. Answer at least one question from each of the following groups: {1,2,3}, {4,5,6}, {7,8,9}, {10,11,12}.

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1. Let
$$A = \begin{pmatrix} -1 & 3 & 0 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Find the characteristic and minimal polynomials of A.

(b) Find the Jordan canonical form of A.

(c) Find a non-singular matrix P such that $P^{-1}AP$ is in Jordan canonical form.

2.(a) Define "Hermitian matrix".

(b) Prove that the eigenvalues of a Hermitian matrix are all real.

(c) What can be said about the eigenvectors corresponding to distinct eigenvalues of a Hermitian matrix? Prove your statement.

3.(a) If V is a vector space of finite dimension n, state the dimensions of the homogeneous parts $\otimes^k V$ and $\Lambda^k V$ of the tensor algebra $\otimes V$ and the exterior algebra ΛV respectively.

(b) If V has dimension 3 and $T: V \to V$ is the linear transformation which has the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ with respect to the basis $\{e_1, e_2, e_3\}$ of V, find the matrix of the induced linear transformation $\Lambda^2 T: \Lambda^2 V \to \Lambda^2 V$ with respect to the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ of $\Lambda^2 V$.

4.(a) Define "solvable group", "nilpotent group", "commutator subgroup".

(b) Prove that the group of nonsingular upper triangular 2×2 matrices over $\mathbb{R}$ is solvable. Is it nilpotent? Why?

5. Suppose that A and B are finite abelian groups with the property that for each natural number k, A and B have the same number of elements of order k. Prove that A is isomorphic to B. (Hint: Use the structure theorem.)

6.(a) State Sylow's Theorems.

(b) Prove that a group of order 35 is cyclic.

7.(a) Prove that $\mathbb{Q}[x]$ has infinitely many prime ideals.

(b) Prove that each nonzero prime ideal of $\mathbb{Q}[x]$ is maximal.

(c) Prove that no principal ideal $(p(x,y))$ of $\mathbb{Q}[x,y]$ is maximal. (Hint: Consider the ideals of the form $(p(x,y),x-a)$.)

8.(a) Define "nilpotent ideal" and "nil ideal" for a ring R.

(b) Prove that all nil ideals of R are contained in the Jacobson radical of R.

(c) Give an example of a ring with a nil ideal which is not nilpotent and justify your example.

9. How many isomorphism classes of indecomposable modules does the ring

$$R = M_2(\mathbb{R}) \times M_2(\mathbb{R}) \times M_3(\mathbb{R})$$

have? Give a brief justification for your answer.

($M_n(\mathbb{R})$ is the ring of all $n \times n$ real matrices.)

10.(a) Find the degrees of the splitting fields K of the following polynomials over the fields k indicated:

(i) x^3-3, $k = \mathbb{Q}$;

(ii) x^3-3, $k = \mathbb{R}$;

(iii) x^p-t, $k = \mathbb{F}_p(t)$.

(Note. In (iii), $\mathbb{F}_p(t)$ is the field of rational functions in one variable t over the finite field $\mathbb{F}_p$ of p elements, p prime.)

(b) Find the automorphism groups $G(K/k)$ of K over k for (i), (ii), (iii) above.

11. Prove that the equation $x^5 - 5x - 2 = 0$ is not solvable by radicals over $\mathbb{Q}$.

12.(a) If $K \subseteq L$ are fields, define the transcendence degree tr.deg.(L/K) of L over K.

(b) If $K \subseteq L \subseteq M$ are fields and tr.deg $(L/K) = m$, tr.deg$(M/L) = n$, what is tr.deg(M/K)? Prove your answer.