UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

The Comprehensive Examination in Algebra

Department of Pure Mathematics

October 30, 1980

TIME: 3 Hours

Answer all questions in part I and 3 questions from part II.

PART I

1a) If c is an element of order mn of a group G where $(m,n) = 1$, show that c can be expressed in one and only one way as a product of two elements a and b which commute and which are of orders m and n respectively.

b) Let a and b be two elements of order m and n respectively in a group G. If a and b both commute with $[a,b] = aba^{-1}b^{-1}$ and $d = (m,n)$, show that $[a,b]^d = 1$.

2a) Let $f(x)$ be an irreducible polynomial of degree greater than 1 with coefficients in a field F. Prove that there is an extension $E \supset F$ such that E contains an element θ satisfying $f(\theta) = 0$.

b) Show that x^3+3 is irreducible over Z_7. Find a field containing Z_7 over which x^3+3 factors completely into linear factors. Determine the degree over Z_7 of the smallest such field.

3. Let V and W be finite-dimensional vector spaces over a field F and let V^* be the dual of V.

(i) Show that there exists a bijective linear map $\Phi: V^* \otimes_F W \to \mathrm{Hom}_F(V,W)$ such that

$$\Phi(f\otimes a)(x) = f(x)a$$

for all $x \in V$, $a \in W$, $f \in V^*$;

3. (con'd)

(ii) Let $e_1,\ldots,e_n$ be a basis of V and $e'_1,\ldots,e'_n$ its dual basis of V^*. Show that the element

$$i = e'_1 \otimes e_1 + e'_2 \otimes e_2 + \ldots + e'_n \otimes e_n \quad \epsilon \quad V^* \otimes_F V$$

is independent of the choice of $e_1,\ldots,e_n$.

4. Let M be a module over a ring R and $f: M \to M$ an endomorphism.

(i) If M is noetherian show that

$$f(M) = M \Rightarrow \ker f = \{0\};$$

(ii) If M is artinian show that

$$\ker f = \{0\} \Rightarrow f(M) = M;$$

(iii) If M is notherian and artinian show that there exist f-invariant submodules M_0 and M_1 such that the restriction of f to M_0 is nilpotent, the restriction of f to M_1 is an automorphism and $M = M_0 \oplus M_1$.

PART II

5. Let T be a linear operator of a finite-dimensional real vector space V.

(i) If $V \neq \{0\}$ show that there exists a T-invariant subspace of V of dimension 1 or 2;

(ii) If T is normal with respect to some inner product in V, show that V is an orthogonal direct sum of T-invariant subspaces of dimension 1 or 2.

6. Let A be a complex $n \times n$ matrix.

   (i) Show that there exist a diagonalizable matrix S and a nilpotent matrix N such that $A = S+N$ and $SN = NS$;

   (ii) Show that S and N in (i) are unique;

   (iii) If A is real show that S and N are real.

7. Let V be a (left) vector space over a division ring D and $R = \mathrm{End}_D(V)$. If W is a subspace of V then $I(W) = \{f \in R: W \subset \mathrm{Ker}\, f\}$ is a left ideal of R.

   (i) If $\dim W = 1$ show that $I(W)$ is a maximal left ideal of R;

   (ii) Show that the Jacobson radical of R is $\{0\}$;

   (iii) If $\dim V < \infty$ show that R is a simple ring.

8. Let J be the Jacobson radical of a ring R and M an R-module.

   (i) If M is simple show that $JM = \{0\}$;

   (ii) If M is finitely generated and $M \neq \{0\}$ show that M has a maximal submodule;

   (iii) If M is finitely generated and $JM = M$ show that $M = \{0\}$.

9. a) Determine the Galois group of x^4+1 over $\mathbb{Q}$, the corresponding splitting field and all the intermediate subfields.

   b) Show that the real roots of x^4-4x+2 are not constructible over $\mathbb{Q}$. (A real number α is constructible over $\mathbb{Q}$ if there exists a finite tower of fields $\mathbb{Q} = K_0 \subset K_1 \subset \dots \subset K_n \subset \mathbb{R}$ such that $[K_i : K_{i-1}] = 2$ and $\alpha \in K_n$.)

10. a) If G is a finite group of order $p^k n$ where p is a prime not dividing n, show that every p-subgroup of G is contained in a subgroup of order p^k.

    b) If G has order 12 and is not isomorphic to A_4, show that G has an element of order 6.