Department of Pure Mathematics

Algebra Comprehensive Examination

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Instructions: Answer seven of the following eight questions. If you answer all eight, clearly indicate which question you do *not* want marked.

Linear Algebra

1. Let
$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
   (a) Show that A is nilpotent of index 3.
   (b) Find the nilpotent matrix M in Jordan canonical form which is similar to A.

2. Let A and B be two $n \times n$ matrices over $\mathbb{C}$.
   (a) Show that if A and B are similar, then they have the same eigenvalues.
   (b) Let A and B be two idempotent matrices, i.e., $A^2 = A$ and $B^2 = B$. Prove that A and B are similar if and only if they are equivalent, i.e., there exist invertible matrices P and Q such that $A = PBQ$.

Group Theory

3. Let G be a finite group of order greater than 2, half of whose elements have order 2, and the other half of the elements form a subgroup of order a power of p, where p is an odd prime.
   (a) Prove that G is solvable.
   (b) Prove that G is not nilpotent (in fact, it has trivial center).
   (c) Give an example of such a group of order 18.

4. (a) State the universal property satisfied by the free group $\mathbb{F}(X)$ generated by a set X.
   (b) Let G be a group with presentation $\langle u, v : uv^2 = v^2u \rangle$.
      i. Prove that G is infinite.
      ii. Prove that G is non-abelian.
      iii. Prove that G has nontrivial center.

Ring Theory

5. Let
$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Q} \right\}$$
be a subring of $M_2(\mathbb{Q})$.

   (a) Describe the Jacobson radical of R.

   (b) Prove that R has precisely two maximal right ideals.

   (c) Prove that R is right Artinian.

   (d) Prove that R has precisely two isomorphism classes of irreducible right modules.

6. Consider the polynomial ring $\mathbb{Z}[x, y, z]$. Let I be the ideal generated by $z^2 - xy$ and let $R = \mathbb{Z}[x, y, z]/I$.

   (a) Prove that R is an integral domain.

   (b) Prove that R is Noetherian.

   (c) Prove that the only units of R are ± 1.

   (d) Prove that R is not a principal ideal domain.

Field and Galois Theory

7. (a) Let E be a field extension of F. Prove that if $[E : F] < \infty$, then E is an algebraic extension of F.

   (b) Let E be a field extension of F. Define
$$L = \{\alpha \in E,\ [F(\alpha) : F] < \infty\}.$$
   Prove that L is a field.

   (c) Determine if the converse of (a) is true. Briefly justify your answer.

8. Let E be the splitting field of $x^3 - 2$ over F.

   (a) Let $F = \mathbb{Q}$.

      i. Compute the Galois group $\mathrm{Gal}_{\mathbb{Q}}(E)$. Justify your answer.

      ii. Write down the lattice of the corresponding intermediate fields of $\mathbb{Q} \subseteq E$.

   (b) Let $F = \mathbb{F}_5$, the finite field of 5 elements.

      i. Compute the Galois group $\mathrm{Gal}_{\mathbb{F}_5}(E)$. Justify your answer.

      ii. Write down the lattice of the corresponding intermediate fields of $\mathbb{F}_5 \subseteq E$.