ANALYSIS & TOPOLOGY Comprehensive Examination

Wednesday, 14 May 2014

Instructions: Attempt all 9 questions. Show all of your reasoning. To pass, you must demonstrate sufficient knowledge in all of the three areas: real analysis, complex analysis, and topology.

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[1] Let X and Y be topological spaces, and let $f : X \to Y$ be a function.

(a) Define the closure (denoted in what follows as "$\mathrm{cl}(A)$") of a subset $A \subseteq X$.

(b) Define what it means for f to be continuous.

(c) Suppose that f is continuous. Prove that $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$, $\forall\, A \subseteq X$.

(d) Suppose that $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$, $\forall\, A \subseteq X$. Does it follow that f is continuous? Justify your answer (proof or counterexample).

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[2] (a) Evaluate the improper integral

$$\int_0^\infty \frac{\sin x}{x(x^2+9)}\,dx.$$

Simplify your answer so that it is expressed in terms of real quantities.

(b) Prove that every root of the polynomial $f(z) = z^6 - 5z^2 + 10$ lies in the annulus $1 < |z| < 2$.

(c) Provide a complete statement of any major theorem used in your solutions to parts (a) or (b).

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[3] Consider the space $X = C([0,1], \mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$. Let $L : X \to \mathbb{R}$ be a function which has the following properties:

- $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$, for all $f, g \in X$ and all α, $\beta \in \mathbb{R}$.
- $L(f \cdot g) = L(f) \cdot L(g)$, for all $f, g \in X$.
- $L(\mathbb{1}) = 1$, where $\mathbb{1} : [0,1] \to \mathbb{R}$ is the function constantly equal to 1.
- $L(\mathbb{F}) = 2/3$, where $\mathbb{F} : [0,1] \to \mathbb{R}$ is defined by $\mathbb{F}(t) = t$, $0 \le t \le 1$.

(a) Let $f, g \in X$ be such that $f(t) \le g(t)$ for every $t \in [0,1]$. Prove that $L(f) \le L(g)$.

(b) Prove that $|L(f)| \le ||f||_\infty$, $\forall\; f \in X$.

(c) Prove that $L(f) = f(2/3)$ for every $f \in X$.

(d) Provide a complete statement of any major theorem used in your solution to (c).

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[4] Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$.

(a) Fix $a \in \mathbb{D}$. Construct a conformal mapping $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi(a) = 0$.

(b) Let f be a holomorphic function on $\mathbb{D}$ such that $|f(z)| < 1$ for all z in $\mathbb{D}$. Suppose also that f has two distinct fixed points. Prove that $f(z) = z$ for all $z \in \mathbb{D}$.

(c) Provide a complete statement of any major theorem used in your solution to (b).

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[5] Let X be a topological space. We will consider the following properties that X may have:

- If every open cover of X has a countable subcover, then X is called a *Lindelöf* space.
- If every countable open cover of X has a finite subcover, then X is called *countably compact.*

(a) State what it means for the space X to be second-countable.

(b) Prove *Lindelöf's lemma*: If X is second-countable, then X is a Lindelöf space.

(c) Suppose X is a Lindelöf space. Prove that X is compact if and only if it is countably compact.

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[6] Let μ be the Lebesgue measure on the interval $[-1, 1]$. Suppose that for every $n \in \mathbb{N}$ we have a non-negative Borel function $f_n : [-1, 1] \to \mathbb{R}$ such that

(i) $\int f_n \, d\mu \;=\; 1$, and

(ii) $f_n(x) = 0$ for every $x \in [-1, 1]$ such that $|x| \geq 1/n$.

(a) Prove that the sequence $(f_n)_{n=1}^{\infty}$ does not have any convergent subsequence in the Banach space $(\, L^1(\mu), || \cdot ||_1 \,)$.

(b) Let $g : [-1, 1] \to \mathbb{R}$ be continuous, and suppose that $g(0) = 14$. Give a brief explanation of why $f_n g$ is integrable for every $n \in \mathbb{N}$, and prove that the sequence of integrals $\left(\int f_n g \, d\mu \right)_{n=1}^{\infty}$ is convergent. What is the limit of this sequence of integrals?

(c) Provide a complete statement of any major theorem you may have used in your solutions to parts (a) or (b).

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[7] Consider an infinite dimensional normed vector space $(X, || \cdot ||)$.

(a) Let Y be a linear subspace of X, such that $Y \neq X$. Prove that $\mathrm{int}(Y) = \varnothing$, where $\mathrm{int}(Y)$ denotes the interior of Y.

(b) Consider the following (true) statement: "If Y is a finite dimensional subspace of X, then $\mathrm{cl}(Y) = Y$." Write one sentence giving the idea of proof of this statement.

(c) In this part of the question we assume that $(X, || \cdot ||)$ is a Banach space. Let S be a subset of X such that $\mathrm{span}(S) = X$, where $\mathrm{span}(S)$ denotes the set of all finite linear combinations of vectors from S. Prove that the set S is uncountable.

(d) Provide a complete statement of any major theorem used in your solution to (c).

(e) Show by example that the conclusion of part (c) no longer holds if we drop the assumption that the normed vector space $(X, || \cdot ||)$ is a Banach space.

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[8] Let D be a bounded open set in $\mathbb{C}$ with closure $\mathrm{cl}(D)$. Let $f : \mathrm{cl}(D) \to \mathbb{C}$ be a continuous function such that the restriction $f|_D : D \to \mathbb{C}$ is holomorphic.

(a) Prove the *minimum modulus principle*: If f is nowhere zero on D, then the minimum of $|f(z)|$ on $\mathrm{cl}(D)$ is attained on the boundary ∂D.

(b) Show by example that the conclusion of part (a) no longer holds if we drop the assumption that f is nowhere zero on D.

(c) Let f be non-constant. Prove that if $|f(z)|$ is constant on ∂D, then f must have at least one zero in D.

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[9] Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from $[-1,1]$ to $\mathbb{R}$.

(a) Define what it means for the sequence $(f_n)_{n=1}^{\infty}$ to be equicontinuous.

(b) Suppose that every f_n is differentiable with $|f_n'(t)| \leq 1$ for every $t \in [-1,1]$. Prove that $(f_n)_{n=1}^{\infty}$ is equicontinuous.

(c) Let $(f_n)_{n=1}^{\infty}$ be the same as in part (b), and assume in addition that $|f_n(0)| \leq 1$ for every $n \in \mathbb{N}$. Prove that there exist a continuous function $f : [-1,1] \to \mathbb{R}$ and indices $1 \leq n(1) < n(2) < \cdots < n(k) < \cdots$ such that the subsequence $(f_{n(k)})_{k=1}^{\infty}$ converges uniformly to f.

(d) Provide a complete statement of any major theorem used in your solution to (c).

(e) In the framework of part (c), show by example that the limit function f may not be differentiable.