Analysis and Topology Comprehensive Exam 2017

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Instructions: Attempt *all ten* questions. The questions are all of equal value.

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[1] Let C be the counterclockwise unit circle centred at the origin. Show that

$$\int_C e^{z+\frac{1}{z}} dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

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[2] Prove that every continuous real-valued function on the square $[0,1] \times [0,1]$ can be uniformly approximated by polynomials in the functions $f(x,y) = x + e^y$ and $g(x,y) = x - e^y$.

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[3] Let $I \subseteq \mathbb{R}$ be an open interval. Let $f : \mathbb{R} \times I \to \mathbb{R} \cup \{\pm\infty\}$. Suppose that f satisfies the following three properties:

[i] for every $t \in I$, the function $x \mapsto f(x,t)$ is Lebesgue integrable;

[ii] for almost all $x \in \mathbb{R}$, the function $t \mapsto f(x,t)$ is finite and is *differentiable* on I with respect to t;

[iii] there exists a Lebesgue integrable function $F : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that for almost all $x \in \mathbb{R}$ and for all $t \in I$, we have $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq F(x)$.

Prove that the function $g(t) = \int_{\mathbb{R}} f(x,t)dx$ is *differentiable* on I and that $g'(t) = \int_{\mathbb{R}} \frac{\partial f}{\partial t}(x,t)dx$.

[*Hint: Use both the mean value theorem and the Lebesgue dominated convergence theorem.*]

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[4] Consider the Hilbert space $\ell^2(\mathbb{N})$. Let $y = (1,1,1,0,\ldots)$, and define a sequence of vectors in $\ell^2(\mathbb{N})$ by $x_1 = (1,2,0,\ldots)$, $x_2 = (0,1,2,0,\ldots)$, $x_3 = (0,0,1,2,0,\ldots),\ldots$

Prove that y is not in the closed linear span of the set $\{x_n : n \in \mathbb{N}\}$.

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[5] Let $C[0,1]$ denote the space of *continuous real-valued functions* on $[0,1]$. Given $f \in C[0,1]$, let $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ denote the uniform norm of f. Let $0 < \alpha < 1$. Define

$$B^\alpha = \Big\{ f \in C[0,1]\,;\, \|f\|_\infty + \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < 1 \Big\}.$$

Prove that any sequence (f_k) in B^α admits a subsequence converging in $C[0,1]$ with respect to the uniform norm. [*Give a precise statement of any major theorem you use.*]

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[6] Given two sets we let $A\Delta B = (A \cup B) \setminus (A \cap B)$ denote their symmetric difference. Let $\mathcal{M}$ be the collection of all Lebesgue measurable subsets of $[0,1]$. Given $A \in \mathcal{M}$ we define its equivalence class to be

$$[A] = \{B \in \mathcal{M} : \mu(A\Delta B) = 0\},$$

where μ denotes Lebesgue measure, and let $[\mathcal{M}]$ denote the collection of all equivalence classes.

[a] Prove that $d([A],[B]) = \mu(A\Delta B)$ defines a metric on $[\mathcal{M}]$.

[b] Prove that $([\mathcal{M}], d)$ is a separable metric space.

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[7] Let $U \subseteq \mathbb{C}$ be open. Recall that

$$L^2(U) = \left\{ f : U \to \mathbb{C}\,;\ \|f\|^2 = \iint_U |f(x+iy)|^2 dxdy < \infty \right\}.$$

Define the subspace $L^2_a(U) = \{f \in L^2(U)\,;\ f \text{ is analytic on } U\}$.

[a] Let $f \in L^2_a(U)$, and $z_0 \in U$. Let $0 < r < \operatorname{dist}(z_0, \partial U)$. Prove that

$$|f(z_0)| \leq \frac{1}{\sqrt{\pi} r} \|f\|.$$

[b] Prove that $L^2_a(U)$ is a closed subspace of $L^2(U)$.

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[8] Let $A_n \subseteq \mathbb{R}$ be a sequence of Borel sets. Let $A = \{x \in \mathbb{R}\,;\ x \in A_n \text{ for infinitely many } n\ \}$.

[a] Prove that A is a Borel set.

[b] Let μ denote Lebesgue measure. Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$, then $\mu(A) = 0$.

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[9] Let R be a closed and bounded set in $\mathbb{C}$ whose interior R^{o} is connected. Let f be continuous and complex-valued on R, and analytic and nonconstant on R^{o}.

[a] Suppose that $f(z) \neq 0$ anywhere in R. Show that $|f(z)|$ has a minimum value on R which occurs only on the boundary ∂R of R, and never in the interior.

[b] Show that the condition $f(z) \neq 0$ in part [a] is necessary. That is, show that $|f(z)|$ *can* attain its minimum value at an interior point of R if that minimum value is zero.

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[10] Let $C^1[a, b]$ denote the space of real-valued functions with a continuous derivative on the interval $[a, b]$ (with one-sided derivatives at the endpoints).

[a] Show that $\|f\| = |f(a)| + \sup\{|f'(t)|; a \leq t \leq b\}$ defines a norm on this space and that $C^1[a, b]$ is complete in this norm.

[b] Give a precise statement of the contraction mapping principle and use it to prove that there is a continuously differentiable function f on $[0, \frac{\pi}{2}]$ satisfying $f(0) = 1$ and $5f'(x) = \sin(xf(x))$.
