University of Waterloo
Department of Pure Mathematics
Analysis Comprehensive Examination
9am–noon, June 1, 2011

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Instructions: There are 10 problems on this exam, worth a total of 110 points, on the topics of real analysis, measure theory, complex analysis and topology. Attempt every problem.

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[7] **1.** **(a)** Give the definition of a G_δ-set in a topological space.

(b) Determine if the set of rational numbers $\mathbb{Q}$ is a G_δ-set in $\mathbb{R}$.

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[10] **2.** Use contour integration to evaluate the improper integral

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}\, dx.$$

Make sure to justify your steps.

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[13] **3.** **(a)** Let $f = u + iv$ and $g = p + iq$ be analytic functions defined in a neighbourhood U of the origin in the complex plane $\mathbb{C}$. Assume that $|g'(0)| < |f'(0)|$. Prove that there is a neighbourhood $V \subset U$ of the origin in which the function $h = f + \overline{g}$ is one-to-one.

(b) Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function which satisfies $|f(z)| \le C(1 + |z|)^n$ for z in $\mathbb{C}$, where $C > 0$ is a constant. Show that f is a polynomial of degree $\le n$.

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[10] **4.** **(a)** Prove that every compact subset of a Hausdorff space is closed.

(b) Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then prove that f is a homeomorphism.

(c) Suppose X is a dense subset of a topological space Y. If X is Hausdorff, must the same be true of Y? Prove or give a counter-example.

[15] **5.** Consider the Cantor set

$$C = \left\{ \sum_{k=1}^{\infty} \frac{t_k}{3^k} : t_1, t_2, \ldots \in \{0,2\} \right\} \subset [0,1]$$

with the usual topology.

(a) Show that C is homeomorphic to $C \times C$. [You may wish to use the product space $P = \{0,1\}^{\mathbb{N}}$.]

(b) Define $\psi : C \to [0,1]$ by $\psi\left(\sum_{k=1}^{\infty} \frac{t_k}{3^k}\right) = \sum_{k=1}^{\infty} \frac{t_k}{2^{k+1}}$.

(i) Show that ψ is continuous and surjective.

(ii) Is ψ a homeomorphism? Is it possible to find a homeomorphism $\varphi : C \to [0,1]$? Justify your answer.

(iii) Determine if ψ satisfies the property of *absolute continuity*, as defined below. *Given $\varepsilon > 0$, there is $\delta > 0$ so that for any n, if $a_1 < b_1 < \ldots < a_n < b_n$ in C satisfies $\sum_{j=1}^{n}(b_j - a_j) < \delta$, we have $\sum_{j=1}^{n}|\psi(b_j) - \psi(a_j)| < \varepsilon$.*

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[10] **6.** For each n, let $f_n : [0,1] \to \mathbb{R}$ be a continuous function which satisfies that $f_n(0) = 0$, and f_n is continuously differentiable on $(0,1)$ with $|f_n'(x)| \leq x$ for x in $(0,1)$.

(a) Prove that there exists a subsequence of $(f_n)_{n=1}^{\infty}$ which converges uniformly to a continuous function f.

(b) Must the limit function f, in (a) above, be differentiable on $(0,1)$? Prove, or provide a counter-example.

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[10] **7.** **(a)** Evaluate $\lim_{n\to\infty} \int_0^{3/n} n\cos(t^3)\,dt$. Justify all of your steps.

(b) Let $g_n : [0,1] \to [0,\infty)$, $n = 1,2,\ldots$ be a sequence of measurable functions such that $\lim_{n\to\infty} \int_0^1 g_n = 0$. Is it the case that $\lim_{n\to\infty} g_n(t) = 0$ for almost every t in $[0,1]$? Prove, or provide a counter-example.

[10] 8. Let (X, d) be a metric space. For any subset $A \subset X$ and any real number $\varepsilon > 0$, let

$$B(A, \varepsilon) = \{x \in X \mid d(x, a) < \varepsilon \text{ for some } a \in A\},$$
$$\overline{B}(A, \varepsilon) = \{x \in X \mid d(x, a) \leq \varepsilon \text{ for some } a \in A\}.$$

(a) Prove that $B(A, \varepsilon)$ is an open subset of X.

(b) Prove that if C is compact then $\overline{B}(C, \varepsilon)$ is a closed subset of X.

(c) Does the result in (b), above, remain true if C is *not* assumed to be compact? Prove, or provide a counter-example.

(d) Show that if C is compact, A is closed and $C \cap A = \varnothing$, then there is $\varepsilon > 0$ for which $\overline{B}(C, \varepsilon) \cap \overline{B}(A, \varepsilon) = \varnothing$.

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[15] 9. (a) State the *Stone-Weierstrass Theorem* for $\mathbb{C}$-valued continuous functions on a compact metric space.

(b) Given a Lebesgue measurable set $A \subset [0, 1]$, $1 \leq p < \infty$, and $\varepsilon > 0$, show that there exists a continuous function $g : [0, 1] \to \mathbb{R}$ such that

$$\|\chi_A - g\|_p = \left(\int_0^1 |\chi_A - g|^p \right)^{1/p} < \varepsilon.$$

[For the case where A is a finite union of open intervals, a labelled picture will suffice.]

(c) Suppose $f : [0, 1] \to \mathbb{R}$ is a bounded Lebesgue measurable function for which $\int_0^1 f(t) e^{nt}\, dt = 0$ for $n = 0, 1, 2, \ldots$. Show that $f(t) = 0$ for almost every t in $[0, 1]$.

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[10] 10. Let $f(z) = 3z^{100} - e^z$.

(a) Counting multiplicities, how many zeros z does f have inside the unit circle, i.e. with $|z| < 1$?

(b) How many zeros of f inside the unit circle are simple?