University of Waterloo
Department of Pure Mathematics
Analysis Comprehensive Examination
Thursday January 28, 2010

Instructions: Do TWO problems from each section. Sections have equal weight. If you attempt three problems in a section, then you must clearly mark which two you want marked.

Basic Real Analysis

A1. (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ for $n \geq 1$ be continuous functions, and suppose that

for every $x \in \mathbb{R}$, there exists an $n \geq 1$ such that $f_n(x) \in \mathbb{Q}$.

Prove that for every $c < d$ in $\mathbb{R}$, one can find some numbers $a < b$ in the interval (c, d) and a positive integer n such that the function f_n is constant on (a, b).

(b) Provide a complete statement of any major theorem used in your solution.

A2. (a) Suppose that $f_n : [0, 1] \to \mathbb{R}$ are C^1 functions such that

$$|f_n(x)| + |f_n'(x)| \leq 1 \quad \text{for all} \quad x \in [0, 1].$$

Prove that the sequence $(f_n)_{n \geq 1}$ has a uniformly convergent subsequence.

(b) Provide a complete statement of any major theorem used in your solution.

A3. Let (X, d) be a compact metric space, and let $T : X \to X$ be a function such that

$$d(Tx, Ty) = d(x, y) \quad \text{for all} \quad x, y \in X.$$

(a) Let a be a point in X. Prove that a is a cluster point of the sequence $\{T^n a : n \geq 1\}$.

(b) Prove that the function T is surjective.

(c) Prove that the function T is a homeomorphism.

Complex Analysis

B1. Let $0 < p < 1$. Evaluate

$$\int_0^\infty \frac{x^p}{1+x^2}\,dx.$$

Simplify your answer so that it is expressed in terms of real quantities.

B2. (a) State the Schwarz Lemma.

(b) Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. Suppose that $f : \mathbb{H} \to \mathbb{C}$ is an analytic function such $|f(z)| < 1$ for all $z \in \mathbb{H}$ and that $f(i) = 0$. Prove that $|f(2i)| \leq 1/3$.

(c) Prove that there exists a unique function f satisfying the hypotheses from part (b) such that $f(2i) = i/3$. Find a formula for f.

B3. Let Ω be a non-empty connected open subset of $\mathbb{C}$, and let $f(z)$ be analytic on Ω.

(a) Define what it means for Ω to be *simply connected*.

(b) If Ω is simply connected, prove that is there an analytic function $g(z)$ on Ω such that $g'(z) = f(z)$.

(c) Provide an example of an analytic function f on a non-simply connected domain Ω which does not have a primitive (anti-derivative).

Topology and Set Theory

C1. (a) Prove that every real vector space has a basis.

(b) Let A be an infinite set. Prove that the collection $\mathcal{F}(A)$ of all finite subsets of A has the same cardinality as A.

(c) Let ℓ^∞ denote the Banach space of all bounded real sequences. Prove that the vector space basis of ℓ^∞ has cardinality $2^{\aleph_0}$.
Hint: for $A \subset \mathbb{N}$, let $s_A = (a_i)$ where $a_i = 1$ if $i \in A$ and $a_i = 0$ otherwise. Observe that a finite dimensional subspace of ℓ^∞ contains at most finitely many such vectors.

C2. (a) Define *connected* and *path-connected* for a topological space.

(b) Prove that $[0, 1]$ is connected.

(c) Prove that path-connected implies connected.

(d) Define a set $X \subset \mathbb{R}^2$ as follows. Let

$$V = \{(0, t) : 0 \le t \le 1\} = \{0\} \times [0, 1]$$

and

$$H_n = \{(t, 1/n) : 0 \le t \le 1\} = [0, 1] \times \{\tfrac{1}{n}\} \quad \text{for} \quad n \ge 1.$$

Set $X = V \cup \{(1, 0)\} \cup \bigcup_{n \ge 1} H_n$. Prove that X is connected, but is not path-connected.

C3. Let $X = \{0, 1\}^{\aleph_0}$ and $Y = \{0, 1, 2\}^{\aleph_0}$ where $\{0, 1\}$ and $\{0, 1, 2\}$ have the discrete topology and X and Y are endowed with the product topology.

(a) Give a base for the topology on X.

(b) Define what a *homeomorphism* is.

(c) Prove that X and Y are homeomorphic.

Measure Theory

D1. Let f, f_n for $n \geq 1$ be bounded Lebesgue measurable functions on $[0,1]$. We say that f_n converges to f *in measure* if

$$\lim_{n\to\infty} m(\{x : |f(x) - f_n(x)| > \varepsilon\}) = 0 \quad \text{for all} \quad \varepsilon > 0.$$

Prove that f_n converges to f in $L^1(0,1)$ if and only if f_n converges to f in measure and $\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1$.

D2. Let $u : [0,1] \to [0,\infty)$ be a Lebesgue measurable function. For every $n \geq 1$, define a function f_n by the formula

$$f_n(x) = \frac{u(x)^n}{1 + u(x)^n}.$$

(a) Briefly explain why the integral $\int f_n \, dm$ exists and is finite for every $n \geq 1$.

(b) Prove that $\lim_{n\to\infty} \int f_n \, dm$ exists, and express it in terms of $m(A)$ and $m(B)$, where $A = \{x : u(x) < 1\}$ and $B = \{x : u(x) > 1\}$.

(c) Provide a complete statement of any major theorem used in your argument.

D3. Let X be an infinite uncountable set. Define

$$\mathcal{M} = \{A \subset X : A \text{ is countable or } X \setminus A \text{ is countable}\}$$

("countable subset" includes the empty set and finite subsets of X).

(a) Verify that $\mathcal{M}$ is a σ-algebra of subsets of X.

(b) Let $f : X \to \mathbb{R}$ be a function which is measurable between $(X, \mathcal{M})$ and $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra of $\mathbb{R}$. Prove that there exists a real number a, uniquely determined, such that $f^{-1}\big(\mathbb{R} \setminus \{a\}\big)$ is a countable set.