University of Waterloo
Department of Pure Mathematics
Analysis and Topology Comprehensive Examination
September, 2020

Prepared by Ken Davidson and Michael Rubinstein.

REMINDER: The Comprehensive Exams will be static open book, in the sense that consultation of all static pre-downloaded materials will be allowed. Examples of allowable aids include standard texts, and personal study notes. However, no other aids are permitted. In particular, discussing (over any medium) the exam with any person during the availability period, or using the internet or any other non-static tool to search questions or concepts, is not allowed.

Your submitted solutions must reflect your own understanding of the concepts being tested, in your own words.

When you submit a Comprehensive Exam, you are agreeing to the following Academic Integrity Statement:

INTEGRITY STATEMENT

I declare the following statements to be true:

1. The work I submit here is entirely my own.
2. I have not used any unauthorized aids.
3. I have not discussed and will not discuss the contents of this examination with anyone until after the submission deadline.
4. I am aware that misconduct related to examinations can result in significant penalties, including failing the examination and suspension (this is covered in Policy 71: `https://uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71`)

Instructions: Answer all questions; they are equally weighted.

Basic Real Analysis.

1. Consider the functions $f_n\colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = nxe^{-nx}$. Does the sequence of functions f_n converge uniformly? Explain.

2. Let $f_0(x) = [0,1] \to \mathbb{R}$ be the function $f_0(x) = x$. Define
$$f_n(x) = -1 + \frac{1}{3}\int_0^1 \cos(3x^2 + y^2 + 1) f_{n-1}(y)dy, \quad n \geq 1.$$
Show that $f_n(x)$ converges uniformly to a continuous function on $[0,1]$.

3. Let (X, d) be a compact metric space. Let V be a closed subspace of $C_{\mathbb{R}}(X)$ such that every $f \in V$ is Lipschitz, i.e. there is some C so that $|f(x) - f(y)| \leq Cd(x, y)$ for $x, y \in X$. Prove that V is finite dimensional.
Hint: show that $A_n = \{f \in V : |f(x) - f(y)| \leq nd(x, y)\}$ has interior for some n.
Hence show that the closed unit ball of V is equicontinuous.

Complex Analysis

4. Let $\log(z)$ denote the branch of the logarithm with imaginary part lying in $(-\pi, \pi]$ and define $\sqrt{z} = \exp(\log(z)/2) = \sqrt{r}\exp(i\theta/2)$ for $z = re^{i\theta}$, where $-\pi < \theta \leq \pi$, $r \geq 0$ and $\sqrt{r}$ denotes the positive square root of r.
   (a) For which complex numbers z does $\sqrt{z^2} = z$ hold?
   (b) Explain which step in the following is false, and why :
$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i^2 = -1.$$

5. Let Ω be a simply connected region in the complex plane, and let $a \in \Omega$. Let $f : \Omega \to \Omega$ be an analytic function such that $f(a) = a$. Prove that $|f'(a)| \leq 1$.

6. Use contour integration to evaluate $\displaystyle\int_0^\infty \frac{\sin^3(t)}{t}\, dt$.
Hint: prove and use the identity $\sin^3(t) = \text{Im}(3e^{it} - e^{3it})/4$.

Topology and Set Theory.

7. Let Y be a topological space and $A \subset Y$. Let $C \subset Y$ be connected, with $C \cap A \neq \emptyset$ and $C \cap (Y \setminus A) \neq \emptyset$. Prove that C must contain points of the boundary $\partial A := \overline{A} \setminus \operatorname{int}(A)$ of A.

8. Let A be the open annulus $A = \{z \in \mathbb{C} : \frac{1}{3} < |z| < 3\}$. Let $\mathcal{H}(A)$ denote the space of all holomorphic functions on A endowed with the topology of uniform convergence on compact subsets.

   (a) Provide a countable neighbourhood base about 0 for this topology.

   (b) Show that the set of functions of the form $f(z) = \sum_{k=-n}^{n} a_k z^k$ is dense in $H(A)$.

   (c) Explain why it is not sufficient to use polynomials.

   (d) Show that $H(A)$ is second countable.

9. Show that there is a subset of the real numbers with the cardinality of the first uncountable cardinal. (You may not assume the continuum hypothesis.)

Measure Theory.

10. Evaluate $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{1 + nx^2}{(1 + x^2)^n}\, dx.$

11. Let $g : [0, 1] \to \mathbb{R}$ be continuous and of bounded variation. Determine with proof whether

$$\lim_{n\to\infty} \sum_{m=0}^{n-1} \left|g\left(\tfrac{m+1}{n}\right) - g\left(\tfrac{m}{n}\right)\right|^2$$

exists, and, if so, it's value.

12. Let $f \in L^2(-1, 1)$ and let $1 \leq p \leq r \leq 2$. Show that $f \in L^p(-1, 1) \cap L^r(-1, 1)$ and there is a constant $C_{p,r}$ such that $\|f\|_p \leq C_{p,r}\|f\|_r$.

13. Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Let $f(\theta) = e^{ia\theta}$ for $\theta \in (-\pi, \pi]$. Evaluate $\|f\|_2^2$ in two ways and deduce that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a - n)^2} = \frac{\pi^2}{\sin^2 a\pi}.$$