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# On Irregularities of Distribution in Shifts and Dilations of Integer Sequences. I

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## 1. Introduction

Let $c_0, c_1, c_2, \ldots$ denote effectively computable positive absolute constants. In 1964 [8], see also [9, 10], Roth studied the extent to which a sequence of integers can be well distributed in all arithmetical progressions. Let $N$ be a positive integer and let $\varepsilon_1, \ldots, \varepsilon_N$ be plus or minus ones. A special case of one of Roth's results [8], is that there exist $c_0$ and $c_1$ as above such that if $N > c_0$ then

$$\max_{a,q,t \in \mathbb{Z}^+} \left| \sum_{j=1}^{t} \varepsilon_{a+jq} \right| > c_1 N^{1/4}. \tag{1}$$

In the above sum we take only those terms $\varepsilon_{a+jq}$ for which $1 \leqq a + jq \leqq N$. We shall observe this convention throughout this paper; equivalently we could suppose that $\varepsilon_i = 0$ for $i < 1$ and $i > N$. It follows from (1) that no matter how we partition $\{1, \ldots, N\}$ into two sets there will always be an arithmetical progression lying in $\{1, \ldots, N\}$ which contains at least $c_1 N^{1/4}$ more terms from one set than from the other. In 1977 Sárközy, see Corollary 4 of [12], improving slightly on another result of Roth [9], showed that (1) still holds with the weaker hypothesis that $\varepsilon_1, \ldots, \varepsilon_N$ are complex numbers of absolute value at least one.

Roth suspected, see [4], that for any positive number $\delta$ there exist positive numbers $C_0(\delta)$ and $C_1(\delta)$ such that if $N > C_0(\delta)$ then (1) holds with $c_1 N^{1/4}$ replaced by $C_1(\delta) N^{1/2-\delta}$. Sárközy [3] showed that this is not the case by constructing, for each integer $N$ larger than one, a sequence $\varepsilon_1, \ldots, \varepsilon_N$ of plus and minus ones such that

$$\max_{a,q,t \in \mathbb{Z}^+} \left| \sum_{j=1}^{t} \varepsilon_{a+jq} \right| < c_2 N^{1/3} (\log N)^{2/3}. \tag{2}$$

In 1981 Beck [1] proved, by a non-constructive argument, that for each integer $N$ greater than one there is a sequence $\varepsilon_1, \ldots, \varepsilon_N$ of plus and minus ones for which (2)

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holds with $c_2N^{1/3}(\log N)^{2/3}$ replaced by $c_3N^{1/4}(\log N)^{5/2}$ and so (1) is best possible apart perhaps from a logarithmic factor.

The goal of this series is to generalize Roth's result by studying, for any sequence of complex numbers $\varepsilon_1, \dots, \varepsilon_N$ and any sequence of positive integers $b_1, b_2, \dots$, the quantity

$$\max_{a \in \mathbb{Z}, q, t \in \mathbb{Z}^+} \left| \sum_{j=1}^{t} \varepsilon_{a+b_j q} \right|. \tag{3}$$

We shall first show, by a slight modification of Roth's method, that if the sequence $b_1, b_2, \dots$ is strictly increasing but is not increasing very quickly then we can obtain a lower bound for the quantity in (3) which yields Roth's result (1) in the special case $b_j = j$ for $j = 1, 2, \dots$.

**Theorem 1.** *Let $N$, $L$ and $b_1, \dots, b_t$ be positive integers with $b_1 < b_2 < \dots < b_t \leqq L$ and let $\varepsilon_1, \dots, \varepsilon_N$ be complex numbers. Put $Q = 14L$. Then*

$$\sum_{q=1}^{Q} \sum_{m=1-qL}^{N} \left| \sum_{j=1}^{t} \varepsilon_{m+b_j q} \right|^2 \geqq (t^2/4) \sum_{j=1}^{N} |\varepsilon_j|^2 .$$

Notice that if $\varepsilon_1, \dots, \varepsilon_N$ are of absolute value at least one then $\sum_{j=1}^{N} |\varepsilon_j|^2 \geqq N$ and so as an immediate consequence of Theorem 1 we obtain the following result.

**Corollary 1.** *Let $N$, $L$ and $b_1, \dots, b_t$ be positive integers with $b_1 < b_2 < \dots < b_t \leqq L$ and $L \leqq N^{1/2}$. Let $\varepsilon_1, \dots, \varepsilon_N$ be complex numbers of absolute value at least one. Then*

$$\max_{\substack{1 \leqq q \leqq 14L \\ -14L^2 < m \leqq N}} \left| \sum_{j=1}^{t} \varepsilon_{m+b_j q} \right| \geqq t/(30L^{1/2}) . \tag{4}$$

For any real number $x$ denote the largest integer less than or equal to $x$ by $[x]$. If we take $t = L = [N^{1/2}]$ and $b_j = j$ for $j = 1, \dots, t$ in Corollary 1 we recover (1).

We remark that (4) is trivial if $t \leqq L^{1/2}$, in other words if the sequence $b_1, b_2, \dots$ increases at least as quickly as the sequence of squares. However, in this context the most interesting sequence to consider after the sequence of consecutive integers is the sequence of squares, see for example Corollary 4. We are able to deal with the sequence of squares by means of a different argument from that employed in the proof of Theorem 1. In fact we are able to treat much more general sequences. In particular we can obtain non-trivial lower bounds for (3) for any increasing sequence $b_1, b_2, \dots$ of polynomial growth. However our general results when specialized to the sequence of squares do not yield as precise results as those obtained below and require more complicated arguments for their proof. Further they do not contain Theorem 1 when the sequence $b_1, b_2, \dots$ grows sufficiently slowly. For this reason, we shall deal with general sequences in a sequel to this paper and shall restrict our attention in the balance of this article to the sequence of squares.

For any non-zero integer $n$ denote the number of positive integers which divide $n$ by $\tau(n)$.

**Theorem 2.** *Let $N$ and $K$ be positive integers and let $\varepsilon_1, \dots, \varepsilon_N$ be complex numbers.*

*Put* $T=\max_{1\leq n\leq K^2}\tau(n)$, $Q=[10^4T^3K\log K]$ *and* $Q'=K'=10TK$. *There exists an effectively computable absolute constant* $c_4$ *such that if* $K>c_4$ *then*

$$\sum_{q=1}^{Q}\sum_{m=1-K^2q}^{N}\left|\sum_{x=1}^{K}\varepsilon_{m+x^2q}\right|^2+8T\log K\sum_{q=1}^{Q'}\sum_{m=1-(K')^2q}^{N}\left|\sum_{x=1}^{K'}\varepsilon_{m+x^2q}\right|^2$$
$$\geqq 200T^3K^2\log K\sum_{j=1}^{N}|\varepsilon_j|^2. \tag{5}$$

We remark that it can be shown by means of the Rudin-Shapiro construction, see [11], that the estimate given for the left hand side of inequality (5) is best possible apart from the constant factor 200.

We are able to deduce from Theorem 2 an estimate, which we believe to be best possible up to a constant factor, for one of the quantities $M$ and $M'$ defined below. Put

$$M=\max_{\substack{1\leqq q\leqq Q\\ -QK^2<m\leqq N}}\left|\sum_{x=1}^{K}\varepsilon_{m+x^2q}\right|,$$

and

$$M'=\max_{\substack{1\leqq q\leqq Q'\\ -Q'(K')^2<m\leqq N}}\left|\sum_{x=1}^{K'}\varepsilon_{m+x^2q}\right|.$$

**Corollary 2.** *Let* $\delta$ *be a positive number, let* $N$ *and* $K$ *be positive integers and let* $\varepsilon_1,\ldots,\varepsilon_N$ *be complex numbers. Define* $T$, $Q$, $Q'$, *and* $K'$ *as in the statement of Theorem 2. There exists an effectively computable absolute constant* $c_5$ *and a number* $C_2(\delta)$ *which is effectively computable in terms of* $\delta$ *such that if* $K>c_5$, $N>C_2(\delta)$ *and*

$$K<N^{1/3}\exp(-(1+\delta)(2\log 2\log N)/3\log\log N), \tag{6}$$

*then*

$$\max(M/K^{1/2},M'/(K')^{1/2})>(1/11)\left(N^{-1}\sum_{j=1}^{N}|\varepsilon_j|^2\right)^{1/2}.$$

Let $\varepsilon_1,\ldots,\varepsilon_N$ be complex numbers of absolute value at least one and apply Corollary 2 with

$$K=[N^{1/3}\exp(-(1+\delta)(2\log 2\log N)/3\log\log N)].$$

By Lemma 3, for $N$ sufficiently large in terms of $\delta$, there exists a sequence $m+q$, $m+4q,\ldots,m+t^2q$ with $1\leqq m+q$ and $m+t^2q\leqq N$ satisfying

$$\left|\sum_{x=1}^{t}\varepsilon_{m+x^2q}\right|>t^{1/2}/11,$$

and with

$$N^{1/6}\exp(-(1+\delta)(\log 2\log N)/3\log\log N)<t<N^{1/3}.$$

We may also deduce from Corollary 2 an estimate for the quantity (3) when $b_j=j^2$ for $j=1,2,\ldots$. By again taking

$$K=[N^{1/3}\exp(-(1+\delta)(2\log 2\log N)/3\log\log N)]$$

we obtain immediately the following result.

**Corollary 3.** *Let $\delta$ be a positive real number, let $N$ be a positive integer and let $\varepsilon_1,\ldots,\varepsilon_N$ be complex numbers of absolute value at least one. There is a number $C_3(\delta)$ which is effectively computable in terms of $\delta$ such that if $N>C_3(\delta)$ then*

$$\max_{\substack{1\leq q\leq T^2N^{1/3}\\ 1\leq t\leq N^{1/3}\\ -N\leq m\leq N}}\left|\sum_{x=1}^{t}\varepsilon_{m+x^2q}\right|>N^{1/6}\exp(-(1+\delta)(\log 2\log N)/3\log\log N). \tag{7}$$

We conjecture that we cannot replace the expression on the right hand side of (7) by $N^\theta$ with $\theta>1/6$ even when we take the maximum of the quantity on the left hand side of (7) over all integers $m$ and all positive integers $q$ and $t$. Beck's result [1] seems to support this conjecture.

Let $u$ be a positive integer. We say that a sequence $\varepsilon_1,\varepsilon_2,\ldots$ is periodic with period $u$ if for all positive integers $i$ and $k$ we have $\varepsilon_i=\varepsilon_{i+ku}$. In [12] Sárközy obtained a periodic analogue of Roth's results (1) and [9]. In a similar fashion we shall establish the following periodic analogue of Theorem 2.

**Theorem 3.** *Let $u$ and $K$ be positive integers and let $\varepsilon_1,\varepsilon_2,\ldots$ be an infinite periodic sequence of complex numbers with period $u$. Put $T=\max_{1\leq n\leq K^2}\tau(n)$, $Q=[10^4T^3K\log K]$ and $Q'=K'=10TK$. There exists an effectively computable absolute constant $c_6$ such that if $K>c_6$ then*

$$\sum_{m=1}^{u}\left(\sum_{q=1}^{Q}\left|\sum_{x=1}^{K}\varepsilon_{m+x^2q}\right|^2+8T\log K\sum_{q=1}^{Q'}\left|\sum_{x=1}^{K'}\varepsilon_{m+x^2q}\right|^2\right)\geq 200T^3K^2\log K\sum_{j=1}^{u}|\varepsilon_j|^2.$$

From Theorem 3 we may derive lower bounds for character sums over shifts of the sequence of squares. We remark that Sárközy [12] slightly improved upon results of Linnik and Renyi [6, 7] for character sums over consecutive integers by using his periodic analogue of Roth's theorem [9]; his results have since been somewhat sharpened by Sokolovskii [13] by a different argument.

**Corollary 4.** *Let $\delta$ be a positive real number, let $p$ be a prime number and let $\chi$ be a character modulo $p$. There is a number $C_4(\delta)$ which is effectively computable in terms of $\delta$ such that if $p>C_4(\delta)$ then*

$$\max_{\substack{1\leq m<p\\ 1\leq j<p}}\left|\sum_{x=1}^{j}\chi(m+x^2)\right|>p^{1/2}\exp(-(\log 2+\delta)\log p/\log\log p). \tag{8}$$

It follows from a result of Weil [15] that if $\chi$ is a non-principal character modulo $p$ with $p$ a prime then for any integer $m$ coprime with $p$,

$$\max_{1\leq j<p}\left|\sum_{x=1}^{j}\chi(m+x^2)\right|<c_7p^{1/2}\log p, \tag{9}$$

and thus (8) is close to best possible. To deduce (9) from the work of Weil one must estimate incomplete exponential sums modulo $p$ in terms of complete sums and this is accomplished by a standard argument, see for instance [2] or [16]. Notice that if we take $\chi$ in Corollary 4 to be the Legendre symbol, with the usual convention that $\left(\frac{a}{p}\right)=0$ whenever $p$ divides $a$, we find that

$$\max_{\substack{1\leq m<p \\ 1\leq j<p}}\left|\sum_{x=1}^{j}\left(\frac{m+x^2}{p}\right)\right|>p^{1/2}\exp(-(\log 2+\delta)\log p/\log\log p). \tag{10}$$

We remark that it is easy to show that if $p$ is an odd prime then for any integer $m$ which is coprime to $p$,

$$\sum_{x=1}^{p}\left(\frac{m+x^2}{p}\right)=-1.$$

We are not aware of any lower bounds comparable to (10).

## 2. Preliminary Lemmas

For any real number $x$ denote by $\|x\|$ the distance from $x$ to the nearest integer. Thus $\|x\|=\min\{x-[x],\,[x]+1-x\}$. Further we put $e(x)=e^{2\pi ix}$.

We first record two standard results from the theory of exponential sums. For proofs of these two lemmas we refer to [14, pp. 23, 24].

**Lemma 1.** *Let $U$ and $V$ be integers with $U<V$ and let $\alpha$ be a real number. Then*

$$\left|\sum_{n=U}^{V}e(n\alpha)\right|\leqq\min(V-U,1/(2\|\alpha\|))$$

*(where we put* $\min(V-U,\,1/0)=V-U$*).*

**Lemma 2.** *Let $r$, $a$, $q'$ and $q$ be integers with $0<q'\leqq q$ and $(a,q)=1$ and let $X$ be a real number with $X\geqq 1$. Then for any real number $\alpha$ with $|\alpha-(a/q)|<1/q^2$ we have*

$$\sum_{y=r}^{r+q'}\min(X,1/(2\|y\alpha\|))<4X+q\log q.$$

We also record an upper estimate for $\tau(n)$, the number of positive divisors of $n$. For a proof see [5, Theorem 317].

**Lemma 3.** *Let $\delta$ be a positive real number. There exists a number $C_5(\delta)$, which is effectively computable in terms of $\delta$, such that if $n>C_5(\delta)$ then*

$$\tau(n)<\exp((\log 2+\delta)\log n/\log\log n).$$

## 3. Proof of Theorem 1

For all real $\alpha$, we put

$$S(\alpha)=\sum_{n=1}^{N}\varepsilon_n e(n\alpha),\qquad F(\alpha)=\sum_{j=1}^{t}e(b_j\alpha)$$

and

$$G(\alpha)=\sum_{q=1}^{Q}|F(q\alpha)|^2.$$

Following Roth's method, we shall compare estimates for the integral $J$ given by

$$J=\int_0^1|S(\alpha)|^2G(-\alpha)d\alpha.$$

We first establish an upper bound for $J$. We have

$$J=\sum_{q=1}^{Q}\int_0^1|S(\alpha)F(-q\alpha)|^2d\alpha=\sum_{q=1}^{Q}\int_0^1\left|\sum_{n=1}^{N}\sum_{j=1}^{t}\varepsilon_n e((n-b_jq)\alpha)\right|^2d\alpha$$
$$=\sum_{q=1}^{Q}\int_0^1\left|\sum_m{}'\left(\sum_{j=1}^{t}\varepsilon_{m+b_jq}\right)e(m\alpha)\right|^2d\alpha,$$

where the dash indicates that the sum above is taken over all integers $m$ which can be expressed in the form $n-b_jq$ with $1\leqq n\leqq N$ and $1\leqq j\leqq t$. Thus by Parseval's identity,

$$J=\sum_{q=1}^{Q}\sum_m{}'\left|\sum_{j=1}^{t}\varepsilon_{m+b_jq}\right|^2\leqq\sum_{q=1}^{Q}\sum_{m=1-qL}^{N}\left|\sum_{j=1}^{t}\varepsilon_{m+b_jq}\right|^2.\tag{11}$$

We shall now establish a lower bound for $J$. By Dirichlet's theorem, for any real number $\alpha$ there exist integers $u$ and $v$ with $1\leqq v\leqq Q$, $(u,v)=1$ and

$$|\alpha-(u/v)|<1/(vQ).\tag{12}$$

Certainly we have

$$G(\alpha)\geqq|F(v\alpha)|^2=\left|\sum_{j=1}^{t}e(b_jv\alpha)\right|^2.\tag{13}$$

For every real number $\beta$ we have $|1-e(\beta)|\leqq 2\pi\|\beta\|$ and so

$$\left|t-\sum_{j=1}^{t}e(b_jv\alpha)\right|\leqq\sum_{j=1}^{t}|1-e(b_jv\alpha)|\leqq\sum_{j=1}^{t}2\pi\|b_jv\alpha\|=2\pi\sum_{j=1}^{t}\|b_jv\alpha-b_ju\|.$$

By (12),

$$2\pi\sum_{j=1}^{t}\|b_jv\alpha-b_ju\|\leqq 7\sum_{j=1}^{t}b_j|v\alpha-u|\leqq 7\sum_{j=1}^{t}L/Q=t/2,$$

hence

$$\left|\sum_{j=1}^{t}e(b_jv\alpha)\right|=\left|t-\left(t-\sum_{j=1}^{t}e(b_jv\alpha)\right)\right|\geqq t-\left|t-\sum_{j=1}^{t}e(b_jv\alpha)\right|\geqq t/2.$$

Thus, from (13),

$$G(\alpha)\geqq t^2/4,\tag{14}$$

for all real numbers $\alpha$. Plainly

$$J\geqq\left(\min_\alpha G(\alpha)\right)\int_0^1|S(\alpha)|^2d\alpha.$$

Thus, by Parseval's identity and (14),

$$J \geqq (t^2/4) \sum_{j=1}^{N} |\varepsilon_j|^2, \tag{15}$$

and Theorem 1 follows from (11) and (15).

## 4. Proof of Theorem 2

For all real $\alpha$, we put

$$S(\alpha)=\sum_{n=1}^{N} \varepsilon_n e(n\alpha), \quad F_1(\alpha)=\sum_{x=1}^{K} e(x^2\alpha), \quad F_2(\alpha)=\sum_{x=1}^{K'} e(x^2\alpha),$$

$$G_1(\alpha)=\sum_{q=1}^{Q} |F_1(q\alpha)|^2, \quad G_2(\alpha)=8T\log K \sum_{q=1}^{Q'} |F_2(q\alpha)|^2$$

and

$$G(\alpha)=G_1(\alpha)+G_2(\alpha).$$

Our proof depends upon a comparison of estimates for $J$ where

$$J=\int_0^1 |S(\alpha)|^2 G(-\alpha)d\alpha.$$

We have

$$\begin{aligned} J &= \sum_{q=1}^{Q}\int_0^1 |S(\alpha)F_1(-q\alpha)|^2 d\alpha + 8T\log K \sum_{q=1}^{Q'}\int_0^1 |S(\alpha)F_2(-q\alpha)|^2 d\alpha \\ &= \sum_{q=1}^{Q}\int_0^1 \left|\sum_{n=1}^{N}\sum_{x=1}^{K} \varepsilon_n e((n-x^2q)\alpha)\right|^2 d\alpha + 8T\log K \sum_{q=1}^{Q'}\int_0^1 \left|\sum_{n=1}^{N}\sum_{x=1}^{K'} \varepsilon_n e((n-x^2q)\alpha)\right|^2 d\alpha \\ &= \sum_{q=1}^{Q}\int_0^1 \left|\sum_m{}^{*}\left(\sum_{x=1}^{K} \varepsilon_{m+x^2q}\right)e(m\alpha)\right|^2 d\alpha + 8T\log K \sum_{q=1}^{Q'}\int_0^1 \left|\sum_m{}'\left(\sum_{x=1}^{K'} \varepsilon_{m+x^2q}\right)e(m\alpha)\right|^2 d\alpha, \end{aligned}$$

where $\sum_m^*$ indicates the summation over all integers $m$ such that $m=n-x^2q$ with $1 \leqq n \leqq N$ and $1 \leqq x \leqq K$ and $\sum_m'$ indicates the summation over all integers $m$ such that $m=n-x^2q$ with $1 \leqq n \leqq N$ and $1 \leqq x \leqq K'$. Therefore, by Parseval's identity

$$J \leqq \sum_{q=1}^{Q} \sum_{m=1-qK^2}^{N} \left|\sum_{x=1}^{K} \varepsilon_{m+x^2q}\right|^2 + 8T\log K \sum_{q=1}^{Q'} \sum_{m=1-q(K')^2}^{N} \left|\sum_{x=1}^{K'} \varepsilon_{m+x^2q}\right|^2. \tag{16}$$

We shall now derive a lower bound for $J$. As a first step we shall establish a lower bound for $\min_\alpha G(\alpha)$. Put $R=4.4(K')^2$. By Dirichlet's theorem, for any real number $\alpha$ there exist integers $u$ and $v$ with $1 \leqq v \leqq R$, $(u,v)=1$ and

$$|\alpha-(u/v)|<1/(vR). \tag{17}$$

Let $\alpha$ be a real number for which (17) holds with

$$K'<v \leqq R. \tag{18}$$

Then

$$G(\alpha)\geqq G_1(\alpha)=\sum_{q=1}^{Q}|F_1(q\alpha)|^2=\sum_{q=1}^{Q}\left(\sum_{x=1}^{K}e(x^2q\alpha)\right)\left(\sum_{y=1}^{K}e(-y^2q\alpha)\right)$$
$$=\sum_{x=1}^{K}\sum_{y=1}^{K}\sum_{q=1}^{Q}e((x^2-y^2)q\alpha)$$
$$=KQ+\sum_{\substack{1\leqq x,y\leqq K\\ x\neq y}}\sum_{q=1}^{Q}e((x^2-y^2)q\alpha)$$
$$=KQ+\sum_{0<|n|<K^2}\left(\sum_{\substack{1\leqq x,y\leqq K\\ x^2-y^2=n}}1\right)\sum_{q=1}^{Q}e(qn\alpha).$$

Since $\sum_{\substack{1\leqq x,y\leqq K\\ x^2-y^2=n}}\leqq\tau(n)$ we see that

$$G(\alpha)\geqq KQ-\sum_{0<|n|<K^2}\tau(n)\left|\sum_{q=1}^{Q}e(qn\alpha)\right|$$
$$\geqq KQ-2T\sum_{n=1}^{K^2}\left|\sum_{q=1}^{Q}e(qn\alpha)\right|.$$

Therefore, by Lemma 1,

$$G(\alpha)\geqq KQ-2T\sum_{n=1}^{K^2}\min(Q,1/(2\|n\alpha\|))$$
$$\geqq KQ-2T\sum_{j=0}^{[K^2/v]}\sum_{n=jv+1}^{(j+1)v}\min(Q,1/(2\|n\alpha\|)).$$

Since $v\leqq R$ we have, by (17), $|\alpha-(u/v)|\leqq 1/v^2$. Thus, by Lemma 2,

$$G(\alpha)\geqq KQ-2T([K^2/v]+1)(4Q+v\log v)$$
$$\geqq KQ-2T(4K^2Q/v+4Q+K^2\log v+v\log v).$$

Since $R=4.4(K')^2$ we have, by (18),

$$G(\alpha)\geqq KQ(1-(8TK/K')-(8T/K)-(2TK\log(4.4(K')^2))/Q$$
$$-(8.8T(K')^2\log(4.4(K')^2))/KQ).$$

On recalling that $K'=10TK$ and using Lemma 3 we see that for $K$ sufficiently large

$$G(\alpha)\geqq KQ/49. \tag{19}$$

Now suppose that $\alpha$ is a real number for which (17) holds with $1\leqq v\leqq K'$. Then plainly

$$G(\alpha)\geqq G_2(\alpha)=8T\log K\sum_{q=1}^{Q'}|F_2(q\alpha)|^2$$
$$\geqq 8T\log K|F_2(v\alpha)|^2=8T\log K\left|\sum_{x=1}^{K'}e(x^2v\alpha)\right|^2. \tag{20}$$

As in the proof of Theorem 1 we may employ the inequality $|1-e(\beta)|\leqq 2\pi\|\beta\|$ to deduce that

$$\left|K'-\sum_{x=1}^{K'} e(x^2 v\alpha)\right| \leqq 2\pi\sum_{x=1}^{K'} x^2|v\alpha-u|<(2\pi/R)\sum_{x=1}^{K'} x^2 .$$

Thus, for $K$ sufficiently large,

$$\left|K'-\sum_{x=1}^{K'} e(x^2 v\alpha)\right| <2.2(K')^3/R=K'/2 .$$

Therefore

$$\left|\sum_{x=1}^{K'} e(x^2 v\alpha)\right| \geqq K'-\left|K'-\sum_{x=1}^{K'} e(x^2 v\alpha)\right| >K'/2 ,$$

and consequently, by (20),

$$G(\alpha)\geqq(8T\log K)(K'/2)^2=200T^3K^2\log K . \tag{21}$$

A comparison of (19) and (21) reveals that (21) holds for all real numbers $\alpha$. Thus, by Parseval's identity and (21),

$$\begin{aligned} J&\geqq\left(\min_{\alpha} G(\alpha)\right)\int_0^1 |S(\alpha)|^2 d\alpha \\ &\geqq(200T^3K^2\log K)\sum_{j=1}^{N} |\varepsilon_j|^2 . \end{aligned} \tag{22}$$

Theorem 2 now follows from (16) and (22).

## 5. Proof of Corollary 2

By Theorem 2 we have

$$\sum_{q=1}^{Q}\sum_{m=1-K^2q}^{N} M^2+8T\log K\sum_{q=1}^{Q'}\sum_{m=1-(K')^2q}^{N}(M')^2\geqq 200T^3K^2\log K\sum_{j=1}^{N}|\varepsilon_j|^2 . \tag{23}$$

For $N>C_2(\delta)$ we have, by (6) and Lemma 3, $K^2Q\leqq 10^4T^3K^3\log K<N$ and $(K')^2Q'\leqq 10^3T^3K^3<N$. Thus, from (23),

$$2QM^2+16(T\log K)Q'(M')^2\geqq 200T^3K^2\log K\left(N^{-1}\sum_{j=1}^{N}|\varepsilon_j|^2\right),$$

and so

$$100M^2/K+8(M')^2/K'>N^{-1}\sum_{j=1}^{N}|\varepsilon_j|^2 .$$

Our result now follows directly.

## 6. Proof of Theorem 3

We shall derive Theorem 3 from Theorem 2. Put $N=Hu$ where $H$ is a positive integer and apply Theorem 2 to the finite sequence $(\varepsilon_1, \ldots, \varepsilon_N)$. Then (5) holds so

$$\sum_{q=1}^{Q} \sum_{m=1-K^2q}^{N} \left|\sum_{x=1}^{K}{}^{*} \varepsilon_{m+x^2q}\right|^2 + 8T\log K \sum_{q=1}^{Q'} \sum_{m=1-(K')^2q}^{N} \left|\sum_{x=1}^{K'}{}^{*} \varepsilon_{m+x^2q}\right|^2 \geqq (200T^3K^2\log K)H \sum_{j=1}^{u} |\varepsilon_j|^2 \tag{23}$$

where * indicates that the inner sums on the left hand side of inequality (23) are taken over those terms $\varepsilon_{m+x^2q}$ with $1 \leqq m+x^2q \leqq N$. Thus there exists a real number $C$ which is effectively computable in terms of $K$ and $\varepsilon_1, \ldots, \varepsilon_u$ such that the left hand side of inequality (23) is at most

$$\sum_{q=1}^{Q} \sum_{m=1}^{N} \left|\sum_{x=1}^{K} \varepsilon_{m+x^2q}\right|^2 + 8T\log K \sum_{q=1}^{Q'} \sum_{m=1}^{N} \left|\sum_{x=1}^{K'} \varepsilon_{m+x^2q}\right|^2 + C, \tag{24}$$

where now we have no restriction on the inner sums above. The expression (24) is equal to

$$H \sum_{m=1}^{u} \left( \sum_{q=1}^{Q} \left|\sum_{x=1}^{K} \varepsilon_{m+x^2q}\right|^2 + 8T\log K \sum_{q=1}^{Q'} \left|\sum_{x=1}^{K'} \varepsilon_{m+x^2q}\right|^2 \right) + C, \tag{25}$$

Thus, from (23) and (25),

$$\sum_{m=1}^{u} \left( \sum_{q=1}^{Q} \left|\sum_{x=1}^{K} \varepsilon_{m+x^2q}\right|^2 + 8T\log K \sum_{q=1}^{Q'} \left|\sum_{x=1}^{K'} \varepsilon_{m+x^2q}\right|^2 \right) + \frac{C}{H} \geqq 200T^3K^2\log K \sum_{j=1}^{u} |\varepsilon_j|^2$$

and our result now follows on letting $H$ tend to infinity.

## 7. Proof of Corollary 4

We apply Theorem 3 with $u=p$, $K$ the largest positive integer for which $p-1 \geqq 400TK$ and $\varepsilon_n = \chi(n)$ for $n=1,2,\ldots$. We obtain, for $K>c_6$,

$$\sum_{m=1}^{p} \left( \sum_{q=1}^{Q} \left|\sum_{x=1}^{K} \chi(m+x^2q)\right|^2 + 8T\log K \sum_{q=1}^{Q'} \left|\sum_{x=1}^{K'} \chi(m+x^2q)\right|^2 \right) \geqq 200T^3K^2(\log K)(p-1).$$

Thus

$$\begin{aligned}\sum_{m=1}^{p-1} &\left( \sum_{\substack{q=1\\(q,p)=1}}^{Q} \left|\sum_{x=1}^{K} \chi(m+x^2q)\right|^2 + 8T\log K \sum_{\substack{q=1\\(q,p)=1}}^{Q'} \left|\sum_{x=1}^{K'} \chi(m+x^2q)\right|^2 \right) \\ &\geqq 200T^3K^2(\log K)(p-1) - QK^2 - 8T(\log K)Q'(K')^2 - p[Q/p]K^2 \\ &\quad - 8T(\log K)p[Q'/p](K')^2 \geqq 200T^3K^2(\log K)(p-1) \\ &\quad - 2\cdot 10^4T^3K^3\log K - 16\cdot 10^3T^4K^3\log K,\end{aligned}$$

which, since $p-1 \geqq 400TK$, is

$$\geqq 100T^3K^2(\log K)(p-1). \tag{26}$$

For each integer $q$ which is coprime with $p$ let $q^*$ denote the integer with $1 \leqq q^* \leqq p-1$ such that $q^*q \equiv 1 (\mathrm{mod}\, p)$. Then for any integer $q$ coprime with $p$ and any integer $m$

$$\left|\sum_{x=1}^{K} \chi(m+x^2q)\right| = \left|\chi(q^*) \sum_{x=1}^{K} \chi(m+x^2q)\right| = \left|\sum_{x=1}^{K} \chi(q^*m+x^2)\right|.$$

Thus by (26)

$$\sum_{m=1}^{p-1} \left( \sum_{\substack{q=1 \\ (q,p)=1}}^{Q} \left|\sum_{x=1}^{K} \chi(q^*m+x^2)\right|^2 + 8T\log K \sum_{\substack{q=1 \\ (q,p)=1}}^{Q'} \left|\sum_{x=1}^{K'} \chi(q^*m+x^2)\right|^2 \right)$$
$$\geqq 100T^3K^2(\log K)(p-1). \tag{27}$$

Put $M = \max\limits_{\substack{1 \leqq j < p \\ 1 \leqq k < p}} \left|\sum_{x=1}^{j} \chi(k+x^2)\right|$. Then, by (27),

$$QM^2 + 8T(\log K)Q'M^2 \geqq 100T^3K^2 \log K,$$

and so $M \geqq (K/101)^{1/2}$. Therefore, by our choice of $K$ and by Lemma 3, for each positive real number $\delta$ there is a real number $C_4(\delta)$, which is effectively computable in terms of $\delta$ such that if $p > C_4(\delta)$ then

$$M > p^{1/2} \exp(-(\log 2 + \delta) \log p / \log\log p),$$

as required.

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