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A NOTE ON THE PRODUCT OF CONSECUTIVE INTEGERS

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1. INTRODUCTION.

Let $S=\{p_1,\dots,p_r\}$ be a set of prime numbers. For any non-zero integer a we shall denote the largest divisor of a composed solely of powers of primes from S by $[a]_S$. Thus $[a]_S = \prod_{p\in S} |a|_p^{-1}$, where $|a|_p$ denotes the p-adic value of a normalized so that $|p|_p=p^{-1}$. Let n and k be positive integers. It follows from work of MAHLER, see for example Theorem 5, II on page 159 of [2], that given any positive number ε,

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$$(1) \qquad [n(n+1)\dots(n+k)]_S < n^{1+\varepsilon},$$

for n sufficiently large in terms of ε, k and S. Mahler's result is ineffective since its proof depends upon a p-adic version of the Thue-Siegel-Roth theorem. Some years ago, Professor Erdős drew my attention to the problem of finding an effective analogue of Mahler's result and this seems an appropriate occasion to discuss this question. The result which we shall obtain, by means of estimates due to Baker and van der Poorten for linear forms in logarithms, is in general much weaker than (1). However if $[n]_S$ is large then we are able to improve upon (1). In addition we shall consider the problem of giving a non-trivial lower bound for $[n(n+1)\dots[n+k)]_S$ which is valid for infinitely many integers n.

Let r be a positive integer. Put $e(1)=e$ and define $e(r)$ inductively for $r=2,3,\dots$ by the rule $e(r) = e^{e(r-1)}$. For any real number x larger than $e(r-1)$ let $\log_r x$ denote the r-fold iterated logarithm of x. Thus $\log_1 x = \log x$ and $\log_r x = \log(\log_{r-1} x)$ for $r=2,3,\dots$.

THEOREM 1. *Let* $S=\{p_1,\dots,p_r\}$ *where* $p_1,\dots,p_r$ *are* *distinct prime numbers with* $r>1$. *There exists a positive integer* k, *which is effectively computable in terms of* $p_1,\ldots,p_r$ *such that for infinitely many positive integers* n,

$$(2)\qquad [n(n+1)\ldots(n+k)]_S > n\log n\ldots\log_{r-1}n.$$

Further, for infinitely many positive integers n

$$(3)\qquad [n(n+1)]_S > c_1 n\log n,$$

while if n *or* $n+1$ *is a power of a prime from* S *then*

$$(4)\qquad [n(n+1)]_S < c_2 n\log n,$$

where c_1 *and* c_2 *are positive numbers which are effectively computable in terms of* $p_1,\ldots,p_r$.

We suspect that estimates (2) and (3) are close to best possible.

We shall now turn to estimates from above for $[n(n+1)\ldots(n+k)]_S$.

THEOREM 2. *Let* $S=\{p_1,\ldots,p_r\}$ *where* $p_1,\ldots,p_r$ *are* *distinct prime numbers. Let* n *and* k *be positive integers with* $n>1$ *and put* $T = 1 + \min_{0\le i\le k} \{(n+i)/[n+i]_S\}$ *and* $t = \min\{k+1, r\}$. *Then*

$$(5) \qquad [n(n+1)\dots(n+k)]_S <$$

$$< \min\{c_0 n^{t-c_1}, \, n\left(\frac{2\log n}{\log T}\right)^{c_2 \log T}\},$$

where c_0, c_1 *and* c_2 *are positive numbers which are effectively computable in terms of* $p_1,\dots,p_r$ *and* k.

The first term in the brackets on the right-hand side of inequality (5) improves upon the trivial estimate $c_0 n^t$ and is obtained by using an estimate due to BAKER [1] for linear forms in logarithms. The second term follows from a p-adic estimate for linear forms in logarithms due to VAN DER POORTEN [3]. We note that if T is a fixed positive number then

$$[n(n+1)\dots(n+k)]_S < n(\log n)^{c_3},$$

where c_3 is a positive number effectively computable in terms of $T, p_1,\dots,p_r$ and k. Plainly this sharpens the result of Mahler. In fact, whenever T is less then

$n^{\varepsilon(n)}$, for any real valued function $\varepsilon(n)$ which tends to zero as n tends to infinity, Theorem 2 yields a stronger lower estimate than (1) for n sufficiently large. Thus we are in the somewhat paradoxical situation that if one integer $n+i$ is divisible by very high powers of primes from S then we obtain a stronger upper estimate for $[n(n+1)\dots(n+k)]_S$ than when this is not the case.

2. PRELIMINARIES.

Let $a_1,\dots,a_n$ be non-zero rational numbers with heights $A_1,\dots,A_n$ respectively and put $A=\max\{4, A_n\}$. By the height of a non-zero rational number r we shall mean the maximum of $|a|$ and $|b|$ where $r=a/b$ and $(a,b)=1$. Let $b_1,\dots,b_{n-1}$ be rational integers with absolute values at most B and put

$$\Lambda = b_1 \log a_1 + \dots + b_{n-1} \log a_{n-1} - \log a_n.$$

In 1973 Baker proved:

LEMMA 1. *If* $\Lambda \neq 0$ *then for any* $\delta > 0$,

$$\log|\Lambda| > \min\{-\delta B, -c_1 \log A\},$$

where c_1 *is a positive number which is effectively computable in terms of* $n, A_1, \ldots, A_{n-1}$ *and* δ.

PROOF. This is Theorem 2 of [1].

In 1977 van der Poorten derived the following p-adic analogue of Lemma 1.

LEMMA 2. *Let* p *be a rational prime number and let* δ *be a real number with* $0<\delta<1$. *If* $a_1^{b_1} \ldots a_{n-1}^{b_{n-1}} a_n^{-1} - 1 \neq 0$ *then*

$$\operatorname{ord}_p(a_1^{b_1} \ldots a_{n-1}^{b_{n-1}}\, a_n^{-1} - 1) <$$

$$< \max\{\delta B, C \log(\delta^{-1} C) \log A\}$$

where C *is a positive number which is effectively computable in terms of* $A_1, \ldots, A_{n-1}, p$ *and* n.

PROOF. This is Theorem 4 of [3].

Let q be a prime number and put

$$t_n = \frac{q^n - 1}{q-1},$$

for $n=1,2,\ldots$.

LEMMA 3. *If* p *is a prime number different from* q *then* p *divides* t_n *for some positive integer* n. *If* l *is the smallest positive integer for which* p *divides* t_l *then*

$$p \geq l.$$

PROOF. We remark that if p divides $q-1$ then p divides $\frac{q^p-1}{q-1}$. Our result now follows from Fermat's theorem.

LEMMA 4. *Let* p *be a prime number different from* q, *let* l *be the smallest positive integer for which* p *divides* t_l *and let* n *be a positive integer. If* l *does not divide* n *then*

$$|t_n|_p = 1,$$

while if $n=kl$ *for some integer* k *then*

$$|t_n|_p = |t_l|_p \, |k|_p.$$

PROOF. This follows from Lemma 8 of [4] on noting that if $p=2$ then $l=2$ and $2|t_4/t_2| = 1$.

3. THE PROOF OF THEOREM 1.

We shall first establish (3). Let l be the smallest positive integer for which p_2 divides $(p_1^l-1)/(p-1)$. By Lemma 3 l exists and is at most p_2. Put

$$n+1 = p_1^{lp_2^m},$$

for some positive integer m. Then, by Lemma 4,

$$|n|_{p_2} = |p_1^l-1|_{p_2}|p_2^m|_{p_2} \leq p_2^{-(m+1)},$$

hence $|n|_{p_2}^{-1}$ is at least $(\log(n+1))/(\log p_1)$. Letting m run through the positive integers we find that (3) holds with $c_1 = (\log p_1)^{-1}$.

If n or $n+1$ is of the form p_i^m for some positive integer m then, since $p_i^m + 1 = (p_i^{2m}-1)/(p_i^m-1)$, (4) follows directly from Lemma 4.

We shall now establish (2). We may assume that p_r is the largest prime in S. Let $l_{i,j}$ denote the smallest positive integer such that p_j divides $(p_i^{l_{i,j}}-1)/(p_i-1)$, $1\leq i<j\leq r$, and put $L_i = l_{i,i+1}\cdots l_{i,r}$

for $i=1,\dots,r-1$. Further, put

$$b_i = p_1^{L_1 p_2^{\cdot^{\cdot^{\cdot^{L_{i-1} p_i^{L_i}}}}}}, \quad \text{for} \quad i=1,2,\dots,r-2.$$

Let $k = b_{r-2} + r^2 p_r$ and for some positive integer m put

$$(6) \qquad n+k = p_1^{L_1 p_2^{\cdot^{\cdot^{\cdot^{L_{r-1} p_r^{m}}}}}} .$$

From Lemma 4 we find that

$$(7) \qquad |n+k-1|_{p_2}^{-1} \geq \frac{\log(n+k)}{L_1 \log p_1} .$$

Furthermore,

$$n + k - b_i =$$

$$= p_1^{L_1 p_2^{\cdot^{\cdot^{\cdot^{L_{i-1} p_i^{L_i}}}}}} \left(p_1^{L_1 p_2^{\cdot^{\cdot^{\cdot^{L_{i-1} p_i^{L_i}}}}}} \times \right.$$

$$\left. \times \left(p_2^{\cdot^{\cdot^{\cdot^{L_i \left(p_{i+1}^{L_{i+1} p_{i+2}^{\cdot^{\cdot^{\cdot^{m}}}}} -1 \right)}}}} \cdot \cdot \cdot -1 \right) -1 \right).$$

and repeated application of Lemma 4 reveals that

$$p_{i+2}^{L_{i+2} p_{i+3}^{\cdot^{\cdot^{\cdot^{L_{r-1} p_r^{m}}}}}}$$

divides $n+k-b_i$, for $i=1,\ldots,r-2$. Thus

$$(8) \qquad |n+k-b_i|_{p_{i+2}}^{-1} \geq \frac{\log_{i+1} n}{2\, L_{i+1} \log p_{i+1}} ,$$

for $i=1,\ldots,r-2$, when m is sufficiently large. By Lemma 3, L_i is at most $p_{i+1} \cdots p_r$ for $i=1,\ldots,r$, whence

(9) $$2^{r-2}L_1 \log p_1 \cdots L_{r-1} \log p_{r-1} \le p_r^{r^2}.$$

Certainly

(10) $$|(n+k-b_{r-2}-1)\cdots(n+k-b_{r-2}-r^2p_r)|_{p_r}^{-1} \ge p_r^{r^2}.$$

Since

$$[n(n+1)\cdots(n+k)]_S \ge$$

$$\ge |n+k|_{p_1}^{-1}|n+k-1|_{p_2}^{-1} \times$$

$$\times |n+k-b_1|_{p_3}^{-1} \cdots |n+k-b_{r-2}|_{p_r}^{-1} \times$$

$$\times |(n+k-b_{r-2}-1)\cdots(n+k-b_{r-2}-r^2p_r)|_{p_r}^{-1},$$

it follows from (6), (7), (8), (9) and (10) that (2) holds for all sufficiently large integers m, as required.

4. THE PROOF OF THEOREM 2.

We shall denote by $c_0, c_1, \ldots$ positive numbers which are effectively computable in terms of $p_1, \ldots, p_r$ and k

only. To each prime p_i in S we associate an integer $n+a_i$ which is divisible by as high a power of p_i as possible subject to the constraint $n \le n+a_i \le n+k$. It is easily seen that if p_i^h exactly divides $n\ldots(n+k)/(n+a_i)$ then p_i^h divides $k!$. Thus

$$(11) \qquad [n\ldots(n+k)]_S \le k! \ |n+a_1|_{p_1}^{-1}\ldots|n+a_r|_{p_r}^{-1}.$$

If $a_1=\ldots=a_r$ our result follows immediately. Accordingly we may assume that $a_1<a_2$.

Since the number of distinct integers in $\{a_1,\ldots,a_r\}$ is at most t, to prove that $[n\ldots(n+k)]_S < c_0 n^{t-c_1}$ it suffices to prove that

$$(12) \qquad [(n+a_1)(n+a_2)]_S < c_3 n^{2-c_4} .$$

To this end we compare estimates for

$$R = \frac{n+a_2}{n+a_1} = 1 + \frac{a_2-a_1}{n+a_1} .$$

Since $0 < \log(1+x) < x$ for x positive, we have

$$0 < \log R < k/n.$$

Certainly we may assume that $n>k^2$, hence

(13) $\log|\log R| < -(\log n)/2.$

Put $u = (n+a_1)/[n+a_1]_S$ and $v = (n+a_2)/[n+a_2]_S$. Then

$$\log R = g_1 \log p_1 + \ldots + g_r \log p_r - \log(u/v),$$

where $g_1,\ldots,g_r$ are integers less than $3 \log n$ in absolute value. Taking $\delta = 1/9$ in Lemma 1 we find that

$$\log|\log R| > \min\{-(\log n)/3, \ -c_5 \log A\},$$

where A denotes the maximum of u and v. A comparison with (13) reveals that

$$A > n^{c_6},$$

whence (12) holds as required.

Let b an integer with $0 \le b \le k$ such that $(n+b)/[n+b]_S = T - 1$. Certainly $[n+b]_S \le n+b \le 2n$ hence to complete our proof it suffices, by (11), to show that if a_i is different from b then

$$(14) \qquad |n+a_i|_{p_i}^{-1} \leq \left(\frac{2\log n}{\log T}\right)^{c_7 \log T}.$$

We have

$$|n+a_i|_{p_i} = \left|\frac{n+b}{b-a_i} - 1\right|_{p_i} |b-a_i|_{p_i},$$

and $n+b = p_1^{l_1} \cdots p_r^{l_r}(T-1)$ where $l_1, \ldots, l_r$ are non-negative integers less than $3 \log n$. If A denotes the maximum of 4 and the height of $(T-1)/(b-a_i)$ then $\log A$ is less then $c_8 \log T$ and thus, on taking $\delta = (\log T)/(2 \log n)$ in Lemma 2 we find that

$$\left|\frac{n+b}{b-a_i} - 1\right|_{p_i}^{-1} = \left|p_1^{l_1} \cdots p_r^{l_r}\left(\frac{T-1}{b-a_i}\right) - 1\right|_{p_i}^{-1} \leq$$

$$\leq \left(\frac{2\log n}{\log T}\right)^{c_9 \log T}.$$

Since $|b-a_i|_{p_i}^{-1}$ is at most k, (14) holds and this establishes the theorem.

REFERENCES

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Note added in proof:

J.C. Lagarias informed me at the conference that he used a similar construction to that given in Theorem 1 of this note to prove a related result in his article "A complement to Ridout's p-adic generalization of the Thue-Siegel-Roth Theorem", Lecture Notes in Math. 899, Springer-Verlag (1981), 264-275.

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