Pure Mathematics Groups and Rings Qualifying Examination
University of Waterloo
September 29, 2022

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let G be a group and let $Z(G)$ be its center. Prove or disprove the following.

   (a) If $G/Z(G)$ is cyclic, then G is abelian.

   (b) If $G/Z(G)$ is abelian, then G is abelian.

   (c) If G is of order p^2, where p is a prime, then G is abelian.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. Let H and K be normal subgroups of a finite group G.

   (a) Show that there exists a one-to-one homomorphism

$$\varphi : G/(H \cap K) \to G/H \times G/K$$

   (b) Show that φ is an isomorphism if and only if $G = HK$.

   (c) Use the second to show that if $\gcd(m, n) = 1$, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let R be the ring $\mathbb{F}_3[x]/\langle x^2 - 1\rangle$, where $\mathbb{F}_3$ is the field $\mathbb{Z}/3\mathbb{Z}$. Show that R is isomorphic to the ring $\mathbb{F}_3 \oplus \mathbb{F}_3$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

4. Let $R = \mathbb{Z}[\sqrt{5}]$. Let $M = (2, 1+\sqrt{5})$ be the R-module generated by 2 and $1+\sqrt{5}$, and let $N = (4, 2+2\sqrt{5})$ be the R-submodule of M generated by 4 and $2+2\sqrt{5}$. Prove that R/M is not isomorphic to M/N as R-modules.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

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