On prime factors of subset sums

by

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1 Introduction

For any set X let $|X|$ denote its cardinality and for any integer n larger than one let $\omega(n)$ denote the number of distinct prime factors of n and let $P(n)$ denote the greatest prime factor of n. In 1934 Erdös and Turán [6] proved that there exists an effectively computable positive constant C_1, such that for any non-empty finite set A of positive integers

$$\omega\left(\prod_{a,a'\in A}(a+a')\right) > C_1 \log |A|. \tag{1}$$

In 1986 Györy, Stewart and Tijdeman [9] , [16] generalized this result to the case where the summands a and a' in (1) are taken from two different sets.

By (1) and the prime number theorem there exists an effectively computable positive constant C_2 such that if A is a finite set of positive integers with $|A| > 1$ then there exist integers a_1, a_2 in A for which

$$P(a_1 + a_2) > C_2 \log |A| \log\log |A|. \tag{2}$$

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It is possible to strengthen estimates (1) and (2) when A is a dense set of integers and in such a case it is even possible to prove the existence of sums which are divisible by large powers of primes. Results of this character, together with their generalizations to the case of sums composed of summands taken from several sets, have been the subject of several recent papers, see for instance those by Balog and Sárközy [1], [2], [3], Erdös, Stewart and Tijdeman [5], Pomerance, Sárközy and Stewart [11] and Sárközy and Stewart [13], [14], [15]. The goal of this paper is to study analogous questions for the subset sums of A, that is the sums which can be formed by adding elements of A without repetition.

For any non-empty finite set A of positive integers we denote by $S(A)$ the set of all positive integers of the form

$$\sum_{a \in A} \epsilon_a a,$$

where ϵ_a is taken from $\{0, 1\}$ for all a in A. We define $s(A)$ by

$$s(A) = \prod_{n \in S(A)} n.$$

For every positive integer m with $m \leq |A|$ there is a positive subset sum of A which is divisible by m. To see this let $t = |A|$ and let $a_1, ..., a_t$ be the elements of A. For each integer m with $1 \leq m \leq t$ either one of the subset sums $a_1, a_1 + a_2, ..., a_1 + ... + a_t$ is divisible by m or at least two of them lie in the same congruence class modulo m. Thus, on taking their difference, we obtain a positive subset sum which is divisible by m. Therefore

$$\omega(s(A)) \geq \pi(|A|),$$

where $\pi(x)$ denotes the counting function for the primes. Further, by the prime number theorem, for each $\epsilon > 0$ there exists a number $C_3(\epsilon)$, which is effectively computable in terms of ϵ, such that

$$P(s(A)) > (1 - \epsilon)|A|, \tag{3}$$

provided that $|A| > C_3(\epsilon)$. We conjecture that

$$\frac{P(s(A))}{|A|} \to \infty \quad \text{as} \quad |A| \to \infty, \tag{4}$$

and that

$$\frac{\omega(s(A))}{\pi(|A|)} \to \infty \quad \text{as} \quad |A| \to \infty. \tag{5}$$

Indeed, perhaps there exist positive constants C_4 and C_5 for which

$$P(s(A)) > C_4|A|^2, \tag{6}$$

and

$$\omega(s(A)) > C_5\pi(|A|^2). \tag{7}$$

On taking $A = \{1, ..., n\}$, for $n = 1, 2, ...$, we see that (6) and (7) cannot be improved; these examples may well be extremal for both problems. For any positive integer k and any finite set of positive integers A let $P(k, A)$ denote the largest k-th power of a prime which divides a positive subset sum of A with the understanding that if there is no such divisor then we put $P(k, A) = 1$. Note that $P(1, A) = P(s(A))$. We conjecture that the prime power analogue of (4), (and perhaps (6)), holds, in other words that for each positive integer k,

$$\frac{P(k, A)}{|A|} \to \infty \quad \text{as} \quad |A| \to \infty. \tag{8}$$

Unfortunately, we have not been able to prove conjectures (4) and (8), or even improve upon (3), without any assumption on A. However, we shall show that there is an effectively computable positive constant C_6 such that if A is a subset of t positive integers, all less than $e^{C_6 t}$, then both (6) and (7) hold. Further we shall show that (4) holds as we run over those sets $A = \{a_1, ..., a_t\}$ whose elements have no common divisor larger than 1 and which satisfy

$$\frac{\log\log a_t}{t} \to \infty \quad \text{as} \quad t \to \infty. \tag{9}$$

We are also able to establish (4) and (5) for those sets A for which there is a real number λ with $\lambda > 1$ such that $a_{i+1}/a_i > \lambda$ for $i = 1, ..., t-1$.

Our first result deals with the case when the elements of A are not too sparse and it allows us to determine various sets B which contain a divisor of a term of $S(A)$. For any real number x let $[x]$ denote the greatest integer less than or equal to x.

Theorem 1 *There exists an effectively computable positive number C_0 such that if $n(\geq 2)$ and t are positive integers, A is a subset of $\{1, ..., n\}$*

of cardinality t and B is a subset of $\{1, ..., [(t/C_0)^2]\}$ consisting of pairwise coprime integers none of which divides a member of $S(A)$ then

$$\sum_{b \in B} \frac{\log b}{b^{1/2}} < C_0 \log n. \tag{10}$$

We shall deduce as a consequence of Theorem 1 the following result.

Corollary 1 *Let C_0 be defined as in the statement of Theorem 1. There exists an effectively computable positive number C_7 such that if n and t are positive integers with*

$$C_7 < t \quad \textit{and} \quad n < e^{t/(5C_0^2)}, \tag{11}$$

and A is a subset of $\{1, ..., n\}$ of cardinality t then at least half of the primes between $(t/C_0)^2/2$ and $(t/C_0)^2$ divide $s(A)$.

Thus, provided that (11) holds, both (6) and (7) also hold. We shall prove Theorem 1 by combining Szemerédi's Theorem about the representation of zero in abelian groups with a version of Gallagher's larger sieve. The constant C_0 is twice the constant C_{17} appearing in the statement of Szemerédi's Theorem, which is given here as Lemma 1.

Our next result gives us some information on the conjecture (8). Also it can be used to deduce finer information on small primes dividing subset sums than can be obtained from Theorem 1. In particular it follows from our next theorem, just as Corollary 1 follows from Theorem 1, that for each integer x with $t \leq x \leq 10^{-9}t^2/\log t$ there is a prime p which divides $s(A)$ with $x < p \leq 2x$ provided that n is at most $\exp(t/(10^8 \log t))$ and that t is sufficiently large.

Theorem 2 *There exist effectively computable positive integers C_8 and C_9 such that if $n (\geq 2)$ and t are positive integers with $t \geq C_8$, A is a subset of $\{1, ..., n\}$ of cardinality t and B is a subset of $\{C_9, ..., [10^{-8}t^2/\log t]\}$ consisting of pairwise coprime integers none of which divides a member of $S(A)$ then*

$$t \sum_{b \in B} \frac{1}{b} < 10^7 \log n.$$

For the proof of Theorem 2 we employ a result of Sárközy (Lemma 2) in place of the result of Szemerédi. For this reason we are required to restrict the range of B somewhat compared to Theorem 1. However for most of this range we are able to obtain sharper results. As a consequence we are able to establish (8) as we run over those subsets A of $\{1, ..., n\}$ of cardinality t for which $\log n < t^{1/k-\epsilon}$, where ϵ is any fixed positive real number. In particular we have the following result.

Corollary 2 *Let k be an integer with $k \geq 2$ and let θ and ϵ be positive real numbers with $1/k < \theta < 1/(k-1)$. There exists a number C_{10} which is effectively computable in terms of θ, ϵ and k such that if n and t are positive integers with*

$$C_{10} < t \quad \text{and} \quad n < e^{t^{1-\theta(k-1)-\epsilon}}, \tag{12}$$

and A is a subset of $\{1, ..., n\}$ of cardinality t then for at least half of the primes p between $t^\theta/2$ and t^θ, p^k divides a member of $S(A)$.

Next we shall establish conjectures (4) and (5) under the assumption that the cardinality of $S(A)$ is large.

Theorem 3 *There exists an effectively computable positive constant C_{11} such that if A is a non-empty set of positive integers then*

$$\omega(s(A)) > C_{11} \log |S(A)|. \tag{13}$$

Notice that if $A = \{a_1, ..., a_t\}$ and $a_{i+1}/a_i \geq 2$ for $i = 1, ..., t-1$ then $|S(A)| = 2^t - 1$ and so, by (13), in this case

$$\omega(s(A)) > C_{12}|A|, \tag{14}$$

and

$$P(s(A)) > C_{13}|A| \log |A|, \tag{15}$$

where C_{12} and C_{13} are effectively computable positive constants. Similarly if λ is a real number with $\lambda > 1$ and $a_{i+1}/a_i > \lambda$ for $i = 1, ..., t-1$ then $a_{i+c}/a_i \geq 2$ for $i = 1, ..., t-c$ where $c = [(\log 2)/\log \lambda] + 1$ and so in this case (14) and (15) hold with C_{12} and C_{13} replaced by positive numbers which are effectively computable in terms of λ.

Our next result allows us to establish conjecture (4) when A is restricted to sets of positive integers with no common prime factor and for which the largest element of A grows very quickly with respect to the cardinality of A.

Theorem 4 *Let t be an integer with $t \geq 2$. Let $A = \{a_1, ..., a_t\}$ be a set of t positive integers whose greatest common divisor is one and assume that a_t is the largest element of A. There is an effectively computable positive number C_{14} such that*

$$P(a_1 \cdots a_t(a_1 + a_t) \cdots (a_{t-1} + a_t)) > C_{14} \log\log a_t. \tag{16}$$

Notice that (4) follows from Theorem 4 provided that (9) holds and that the elements of A have no common factor. In [9], Győry, Stewart and Tijdeman proved a closely related result to Theorem 4. Let ϵ be a positive real number and let $a_1, ..., a_t$ be positive integers having no common factor. They proved that there is a number C_{15} which is effectively computable in terms of ϵ and an effectively computable positive constant C_{16} such that if t is greater than C_{15} then

$$P(a_1 \cdots a_{t-1}(a_1 + a_t) \cdots (a_{t-1} + a_t)) > \min((1-\epsilon)t\log t, C_{16} \log\log(a_{t-1} + a_t)).$$

2 Preliminary Lemmas

We shall require the following special case of a result of Szemerédi [17] which generalized earlier work of Erdös and Heilbronn [4].

Lemma 1 *Let b be a positive integer, let A be a set of positive integers and let $\nu_A(b)$ denote the number of residue classes modulo b that contain an element of A. There is an effectively computable positive constant C_{17} such that if*

$$\nu_A(b) > C_{17} b^{1/2}, \tag{17}$$

then there is a member of $S(A)$ which is divisible by b.

For the proof of Theorem 2 we shall employ the following theorem of Sárközy (see Theorem 7 of [12]).

Lemma 2 *Let b be a positive integer and let A be a finite set of positive integers. For each positive integer k we denote by n_k the number of elements of A which are congruent to k modulo b. There exists an effectively computable positive constant C_{18} such that if $b > C_{18}$,*

$$|A| > 2 \cdot 10^3 (b \log b)^{1/2}, \tag{18}$$

and

$$\sum_{k=1}^{b} n_k^2 < \frac{|A|^3}{4 \cdot 10^6 b \log b}, \tag{19}$$

then there is a member of $S(A)$ which is divisible by b.

Notice that if A is a set of integers no two of which are congruent modulo b then

$$\sum_{k=1}^{b} n_k^2 = |A|,$$

and so, provided that (18) holds, (19) is satisfied. In particular we obtain a slightly weaker version of Szemerédi's Theorem with condition (17) replaced by the slightly more stringent condition (18). We believe that the factor $\log b$ is not required in (18) or (19). If it could be eliminated from (18) and (19) then Lemma 1 would be a special case of Lemma 2.

For the proof of Theorem 1 we shall require a modified version of Gallagher's larger sieve [8].

Lemma 3 *Let m and n be positive integers and let A be a subset of $\{m+1, ..., m+n\}$. Let B be a finite set of pairwise coprime positive integers. For each b in B let $\nu(b)$ denote the number of residue classes modulo b that contain an element of A. Then*

$$|A| \leq \frac{\sum_{b \in B} \log b - \log n}{\sum_{b \in B} \frac{\log b}{\nu(b)} - \log n},$$

provided that the denominator of the above expression is positive.

Proof We shall follow Hooley (see page 19 of [10]). Let n_k denote the number of terms of A which are congruent to k modulo b. By the Cauchy-Schwarz inequality

$$|A|^2 = \left(\sum_{k=1}^{b} n_k \right)^2 \leq \nu(b) \sum_{k=1}^{b} n_k^2.$$

Since

$$\sum_{k=1}^{b} n_k^2 = \sum_{a\equiv a' (\bmod b),\ a,a'\in A} 1 = |A| + \sum_{b|(a-a'),\ a\neq a'} 1$$

we have

$$|A|^2 \sum_{b\in B} \frac{\log b}{\nu(b)} \leq |A| \sum_{b\in B} \log b + \sum_{a\neq a'} \sum_{b\in B,\ b|(a-a')} \log b.$$

Since the terms of B are pairwise coprime,

$$\sum_{b\in B,\ b|(a-a')} \log b \leq \log |a-a'|$$

whenever $a \neq a'$. Therefore

$$|A|^2 \sum_{b\in B} \frac{\log b}{\nu(b)} \leq |A| \sum_{b\in B} \log b + (|A|^2 - |A|) \log n,$$

and the result follows directly.

We next state a special case of Evertse's theorem concerning the number of solutions of S-unit equations in two variables, see Corollary 1 of [7]. For any non-zero rational number x and any prime number p there is a unique integer a such that $p^{-a}x$ is the quotient of two integers coprime with p. We say that a is the p-adic order of x. We put $\mathrm{ord}_p x = a$.

Lemma 4 *Let w be positive integer and let S be a set of w prime numbers. The equation*

$$x + y = 1,$$

has at most $3 \cdot 7^{2w+3}$ solutions in pairs (x, y) of non-zero rational numbers which have p-adic order zero for all primes p not in S.

For the proof of Theorem 4 we shall appeal to the following estimate for p-adic linear forms in the logarithms of algebraic numbers due to Yu (see Corollary 2 and Lemma 1.4 of [18]).

Lemma 5 *Let n be a positive integer. Let $a_1, ..., a_n$ be non-zero integers with absolute values at most $A_1, ..., A_n$ respectively and let $b_1, ..., b_n$ be integers of* *absolute values at most B. Assume $A_n \geq A_i \geq 3$, for $i = 1, ..., n$ and $B \geq 3$. Let p be a prime number. If $a_1^{b_1} \cdots a_n^{b_n} - 1 \neq 0$ then*

$$\mathrm{ord}_p\left(a_1^{b_1} \cdots a_n^{b_n} - 1\right) < p^2 (n+1)^{C_{19} n} \log A_1 \cdots \log A_n \ \log\log A_n \log B,$$

where C_{19} is an effectively computable positive constant.

3 Proofs of Theorems

Proof of Theorem 1 If b does not divide $s(A)$ then by Lemma 1, $\nu_A(b) \leq C_{17} b^{1/2}$. Put $\nu(b) = \nu_A(b)$ and take $C_0 = 2C_{17}$. Suppose that (10) fails to hold. Then

$$\sum_{b \in B} \frac{\log b}{\nu(b)} \geq 2 \log n,$$

hence

$$\sum_{b \in B} \frac{\log b}{\nu(b)} - \log n \geq \frac{1}{2} \sum_{b \in B} \frac{\log b}{\nu(b)}.$$

Therefore, by Lemma 3,

$$t = |A| < \frac{\sum_{b \in B} \log b}{\frac{1}{2} \sum_{b \in B} \frac{\log b}{\nu(b)}} \leq 2 \max_{b \in B} \nu(b) \leq \max_{b \in B} C_0 b^{1/2} \leq t,$$

which is a contradiction.

Proof of Corollary 1 We shall suppose that at least half of the primes between $(t/C_0)^2/2$ and $(t/C_0)^2$ do not divide $s(A)$ and we shall show that if (11) holds then this leads to a contradiction. Let B be the set of these primes. There is an effectively computable positive constant C_{20} such that if t is greater than C_{20} then $|B| \geq t^2/(9C_0^2 \log t)$ and

$$\sum_{b \in B} \frac{\log b}{b^{1/2}} \geq \frac{|B| \log((t/C_0)^2)}{t/C_0} > \frac{t}{5C_0}.$$

Thus if (11) holds then we obtain a contradiction by Theorem 1. Our result now follows.

Proof of Theorem 2 Let A be a subset of $\{1, ..., n\}$ of cardinality t and let B be a subset of $\{[C_{18}+1], ..., [10^{-8} t^2 / \log t]\}$ consisting of pairwise coprime integers none of which divides a member of $S(A)$; here C_{18} is the constant specified in Lemma 2. C_{21}, C_{22}, C_{23} will denote positive effectively computable constants.

Put

$$d = d(A) = \prod_{a,a' \in A,\ a>a'} (a - a'),$$

and observe that since A is a subset of $\{1, ..., n\}$,

$$d \le n^{\binom{t}{2}} < n^{t^2/2}. \tag{20}$$

For each integer b in B let $r(b)$ denote that non-negative integer which satisfies

$$b^{r(b)} | d \quad \text{and} \quad b^{r(b)+1} \not| d.$$

For each positive integer k we define n_k to be the number of elements of A which are congruent to k modulo b. Clearly if a and a' are both congruent to k modulo b then b divides $a - a'$. Thus

$$r(b) \ge \sum_{k=1}^{b} \binom{n_k}{2} = \frac{1}{2}\left(\left(\sum_{k=1}^{b} n_k^2\right) - t\right). \tag{21}$$

We now apply Lemma 2. Note that (18) holds provided that $t > C_{21}$. If b is in B then b does not divide any element of $S(A)$ and so, by Lemma 2,

$$\sum_{k=1}^{b} n_k^2 \ge \frac{t^3}{4 \cdot 10^6 b \log b},$$

for $t > C_{21}$. For $t > C_{22}$, $t^3/(4 \cdot 10^6 b \log b)$ is at least $2t$ and, by (21),

$$r(b) \ge \frac{t^3}{1.6 \cdot 10^7 b \log b}, \tag{22}$$

for each element b in B. Since the elements of B are pairwise coprime

$$\prod_{b \in B} b^{r(b)} \le d. \tag{23}$$

Thus, by (20), (22) and (23),

$$\frac{t^2 \log n}{2} \ge \log d \ge \sum_{b \in B} r(b) \log b \ge \frac{t^3}{1.6 \cdot 10^7} \sum_{b \in B} \frac{1}{b},$$

for $t > C_{23}$. Our result now follows.

Proof of Corollary 2 We shall suppose that for at least half of the primes p between $t^\theta/2$ and t^θ, p^k divides no member of $S(A)$ and we shall show that this leads to a contradiction. Let B be the set of k-th powers of these primes.

C_{24} and C_{25} will denote positive numbers which are effectively computable in terms of θ, ϵ and k. B is a subset of $\{[t^{k\theta}/2^k], ..., [t^{k\theta}]\}$ which is contained in $\{C_9, ..., [10^{-8}t^2/\log t]\}$ and

$$|B| > \frac{t^\theta}{3\theta \log t},$$

whenever t is greater than C_{24}. Thus

$$10^{-7}t\sum_{b\in B}\frac{1}{b} > \frac{10^{-7}t^{1+\theta-\theta k}}{3\theta\log t} > t^{1-\theta(k-1)-\epsilon},$$

for $t > C_{25}$. We now appeal to Theorem 2 to obtain a contradiction if (12) holds and so to complete the proof.

Proof of Theorem 3 Put $w = \omega(s(A))$ and let $p_1, ..., p_w$ be the primes which divide $s(A)$. Define h by

$$\sum_{a\in A} a = h.$$

Note that we may express h as a sum $x + y$ with both x and y in $S(A)$ in at least $|S(A)| - 1$ different ways since we may take x to be $\sum_{a\in A}\epsilon_a a$ with ϵ_a in $\{0, 1\}$ for all a in A subject only to the constraint that not all the ϵ_a's are zero and not all the ϵ_a's are one. Since x, y and h are in $S(A)$ their prime factors come from the set $\{p_1, ..., p_w\}$. Thus each pair (x, y) gives a distinct solution $(X, Y) = (x/h, y/h)$ of the equation $X + Y = 1$ in non-zero rational numbers which have p-adic order zero except perhaps for the primes $p_1, ..., p_w$. Therefore, by Lemma 4,

$$|S(A)| - 1 \leq 3 \cdot 7^{2w+3},$$

and our result now follows on taking logarithms.

Proof of Theorem 4 Let $p_1, ..., p_w$ be the primes which divide $a_1 \cdots a_t (a_1 + a_t) \cdots (a_{t-1} + a_t)$. We have

$$a_t = \prod_{i=1}^{w} p_i^{\mathrm{ord}_{p_i} a_t}, \tag{24}$$

and to obtain our result we shall estimate $\mathrm{ord}_{p_i} a_t$ from above for $i = 1, ..., w$. Let P denote the maximum of $p_1, ..., p_w$. $C_{26}, C_{27}, ...$ will denote effectively computable positive constants.

Let i be an integer with $1 \le i \le w$ and suppose that $\mathrm{ord}_{p_i} a_t$ is positive. Then p_i divides a_t and there exists an integer $g = g(i)$ with $1 \le g \le t-1$ for which p_i does not divide a_g since the greatest common divisor of $a_1, ..., a_t$ is one. Therefore

$$\mathrm{ord}_{p_i} a_t = \mathrm{ord}_{p_i}((a_g + a_t) - a_g) = \mathrm{ord}_{p_i}\left(\left(\frac{a_g + a_t}{a_g}\right) - 1\right). \tag{25}$$

We write

$$\frac{a_g + a_t}{a_g} = p_1^{l_1} \cdots p_w^{l_w},$$

where $l_1, ..., l_w$ are integers of absolute value at most $3 \log a_t$. Thus, by (25) and Lemma 5,

$$\mathrm{ord}_{p_i} a_t < p_i^2 (2w)^{C_{26} w} \log p_1 \cdots \log p_w \log\log P \log\log a_t.$$

Certainly

$$(2w)^{C_{26} w} \log p_1 \cdots \log p_w < (2w \log P)^{C_{26} w}.$$

By the prime number theorem $w < C_{27} P / \log P$, and thus

$$\mathrm{ord}_{p_i} a_t < e^{C_{28} P} \log\log a_t,$$

for $i = 1, ..., w$. Therefore, by (24),

$$a_t < P^{w e^{C_{28} P} \log\log a_t}.$$

Since $w < P$ we find, on taking logarithms,

$$\frac{\log a_t}{\log\log a_t} < e^{C_{29} P},$$

hence $P > C_{30} \log\log a_t$ as required.

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