On the representation of an integer in two different bases

By *C. L. Stewart**) at Waterloo

1. Introduction

In 1970 Senge and Strauss [4] proved that the number of integers, the sum of whose digits in each of the bases a and b lies below a fixed bound, is finite if and only if $\frac{\log a}{\log b}$ is irrational. Their proof, which depends upon a generalization due to Mahler of the Thue-Siegel-Roth theorem, is not effective since, given a fixed bound, it does not yield a method for determining the largest integer n for which the sum of the digits of n in each of base a and base b lies below the bound. In this paper we shall exhibit a lower bound for the sum of the digits of n in base a plus the sum of the digits of n in base b which is effectively computable and which tends to infinity as n tends to infinity.

Let a, b and n be integers larger than 1 and let α and β be integers satisfying $0 \leqq \alpha < a$ and $0 \leqq \beta < b$. Denote the numbers of digits in the canonical expansion of n in base a which are different from α by $L_{\alpha,a}(n)$. Define $L_{\beta,b}(n)$ similarly and put

$$L_{\alpha,a,\beta,b}(n) = L_{\alpha,a}(n) + L_{\beta,b}(n).$$

We remark that for all n

$$L_{\alpha,a,\beta,b}(n) < C_1 \log n,$$

and that on average

$$L_{\alpha,a,\beta,b}(n) > C_2 \log n,$$

where C_1 and C_2 are positive numbers which are effectively computable in terms of a and b only. We shall establish the following theorem.

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Theorem 1. *If* $\frac{\log a}{\log b}$ *is irrational then*

$$L_{\alpha,a,\beta,b}(n) > \frac{\log\log n}{\log\log\log n + C} - 1,$$

for $n > 25$, *where* C *is a positive number which is effectively computable in terms of* a *and* b *only.*

Theorem 1 shows that a sufficiently large integer cannot have a simple representation in both base a and base b unless a and b are multiplicatively dependent.

The result of Senge and Strauss follows from Theorem 1 on taking $\alpha = \beta = 0$ and observing that the sum of the digits of n in base a plus the sum of the digits of n in base b is at least the number of non-zero digits of n in base a plus the number of non-zero digits of n in base b. Note that if $\frac{\log a}{\log b}$ is rational, so that $a^r = b^s$ for integers r and s, then the sum of the digits of n in base a plus the sum of the digits of n in base b is 2 for those integers n of the form $a^{rm} = b^{sm}$ for $m = 1, 2, \ldots$, and certainly

$$L_{0,a,0,b}(n) = 2$$

for these n.

Let u_n be the n-th term of a general linear recurrence sequence satisfying

$$u_n = d_1 u_{n-1} + \cdots + d_r u_{n-r},$$

where $d_1, \ldots, d_r$ and $u_1, \ldots, u_r$ are integers. Then

$$u_n = P_1(n)\lambda_1^n + \cdots + P_k(n)\lambda_k^n, \tag{1}$$

where $\lambda_1, \ldots, \lambda_k$ are the roots of the characteristic polynomial associated with u_n. Further $P_1(n), \ldots, P_k(n)$ are polynomials with coefficients from $Q(\lambda_1, \ldots, \lambda_k)$ and degrees the multiplicities of $\lambda_1, \ldots, \lambda_k$ respectively in the characteristic polynomial of u_n; here Q denotes the rational numbers. We shall also prove the following result.

Theorem 2. *Let* a *be an integer larger than* 1 *and let* α *be a non-negative integer less than* a. *If* u_n *satisifes* (1), $|\lambda_1| > \max\{1, |\lambda_2|, \ldots, |\lambda_k|\}$, $P_1(x)$ *is not identically zero and* $\frac{\log \lambda_1}{\log a}$ *is irrational then*

$$L_{\alpha,a}(u_n) > \frac{\log n}{\log\log n + C_0} - 1,$$

for $n > 4$, *where* C_0 *is a positive number which is effectively computable in terms of* $a, \lambda_1, \ldots, \lambda_k, r$ *and the coefficients of* $P_1(x), \ldots, P_k(x)$.

One consequence of Theorem 2 is that the complexity of the radix representation for the n-th Fibonacci number v_n increases with n for any base. In particular, the sum of the digits in the canonical expansion of v_n in base a, for $a > 1$, tends to infinity as n tends to infinity. For we have

$$v_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right),$$

for $n=1, 2, \ldots$ Put $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$ and observe that $\frac{\log \lambda_1}{\log a}$ is irrational for all positive integers a since λ_1^k is irrational for all non-zero integers k. Thus the hypotheses of Theorem 2 are satisfied and the aforementioned result holds.

I should like to thank H. W. Lenstra Jr. for inviting me to his thesis defence since it was on this occasion that I proved Theorem 1.

2. Preliminary lemmas

Let $b_1, b_2, \ldots, b_n$ denote rational integers with absolute values at most B and let $\alpha_1, \ldots, \alpha_n$ denote non-zero algebraic numbers with degrees at most d and heights at most $A_1, \ldots, A_n$ respectively. We assume that B and $A_1, \ldots, A_n$ are all at least 4. By the height of an algebraic number we shall mean the maximum of the absolute values of the relatively prime integer coefficients in the minimal defining polynomial of the number. We set

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n,$$

where the logarithms have their principal values, and

$$\Omega = \log A_2 \cdots \log A_n.$$

In 1976 Baker [1], Theorem 2, proved the following result.

Lemma 1. *If* $\Lambda \neq 0$ *then*

$$|\Lambda| > \exp(-C_3 \log A_1 \Omega \log \Omega \log B),$$

where C_3 *is a positive number which is effectively computable in terms of* n *and* d *only.*

In the same year Loxton and van der Poorten [2], Theorem 1, (see also [2], Lemma 1 and [3]), investigated the degenerate situation when $\Lambda = 0$. They showed that if $\Lambda = 0$ then there exists a non-trivial linear dependence relation among the logarithms with integer coefficients which are small.

Lemma 2. *Assume that* $A_1 \leqq A_2 \leqq \cdots \leqq A_n$, *that* $b_1, \ldots, b_n$ *are not all zero and that* $\alpha_1, \ldots, \alpha_n$ *are positive real numbers. If* $\Lambda = 0$ *then there exists a relation*

$$b_1' \log \alpha_1 + \cdots + b_n' \log \alpha_n = 0,$$

with $b_1', \ldots, b_n'$ *integers, not all of which are zero, satisfying*

$$\max_{1 \leqq i \leqq n} |b_i'| \leqq C_4 \Omega,$$

where C_4 *is a positive number which is effectively computable in terms of* n *and* d *only.*

3. The proof of Theorem 1.

We consider the following two expansions of n, for $n>a+b$:

$$n=a_1a^{m_1}+\alpha\left(\frac{a^{m_1}-1}{a-1}\right)+a_2a^{m_2}+\cdots+a_ra^{m_r},$$

$$n=b_1b^{l_1}+\beta\left(\frac{b^{l_1}-1}{b-1}\right)+b_2b^{l_2}+\cdots+b_tb^{l_t}$$

where

$$0<a_1<a,\ -\alpha\leqq a_i<a-\alpha \quad \text{with} \quad a_i\neq 0 \quad \text{for} \quad i=2,\ldots,r,$$

$$0<b_1<b,\ -\beta\leqq b_i<b-\beta \quad \text{with} \quad b_i\neq 0 \quad \text{for} \quad i=2,\ldots,t,$$

and where

$$m_1>m_2>\cdots>m_r\geqq 0 \quad \text{and} \quad l_1>l_2>\cdots>l_t\geqq 0.$$

We put

$$\theta=c_1\log\log n, \tag{2}$$

where c_1 is a positive number larger than 4 which is effectively computable in terms of a and b only. We shall assume that $n>c_2>25$, where $c_2, c_3, \ldots$ are positive numbers which are computable in terms of a and b only and which may be determined independently of c_1.

We now regard the intervals

$$\Theta_1=(0,\theta],\ \Theta_2=(\theta,\theta^2],\ldots,\ \Theta_k=(\theta^{k-1},\theta^k],$$

where k satisfies the inequalities,

$$\theta^k\leqq\frac{\log n}{4\log a}<\theta^{k+1}. \tag{3}$$

If each interval Θ_s, for $s=1,\ldots,k$, possesses at least one term either of the form m_1-m_i or of the form l_1-l_j the theorem holds since then

$$L_{\alpha,a,\beta,b}(n)\geqq r+t-2\geqq k, \tag{4}$$

while from (3) we have

$$(k+1)\log\theta>\log\log n-\log(4\log a),$$

hence

$$k>\frac{\log\log n}{\log\log\log n+\log c_1}-\log(4\log a)-1. \tag{5}$$

The theorem follows from (4) and (5) since $L_{\alpha,a,\beta,b}(n)$ is always at least zero.

Therefore we may assume that there exists an integer s, with $1 \leqq s \leqq k$, for which Θ_s contains no numbers of the form $m_1 - m_i$ or $l_1 - l_j$. Define p and q by the inequalities

$$m_1 - m_p \leqq \theta^{s-1}, \quad m_1 - m_{p+1} \geqq \theta^s, \tag{6}$$

$$l_1 - l_q \leqq \theta^{s-1}, \quad l_1 - l_{q+1} \geqq \theta^s, \tag{7}$$

with the convention that m_{r+1} and l_{t+1} are zero. We now write

$$\begin{aligned}(b-1)(a-1)n &= ((b-1)(a-1)a_1 + (b-1)\alpha)a^{m_1} \\ &\quad + (b-1)(a-1)a_2 a^{m_2} + \cdots + (b-1)(a-1)a_r a^{m_r} - (b-1)\alpha \\ &= A_1 a^{m_p} + A_2,\end{aligned}$$

where A_1 and A_2 are integers. We have

$$0 < A_1 < (b-1)(a-1)a^{m_1 - m_p + 1} + (b-1)\alpha a^{m_1 - m_p}$$

hence

$$0 < A_1 < 2(b-1)(a-1)a^{m_1 - m_p + 1}. \tag{8}$$

An easy calculation shows that

$$0 \leqq |A_2| < 2(b-1)(a-1)a^{m_{p+1}+1}.$$

Similarly

$$(b-1)(a-1)n = B_1 b^{l_q} + B_2$$

where

$$0 < B_1 < 2(b-1)(a-1)b^{l_1 - l_q + 1}, \tag{9}$$

and

$$0 \leqq |B_2| < 2(b-1)(a-1)b^{l_{q+1}+1}.$$

We have

$$1 = \frac{A_1 a^{m_p} + A_2}{B_1 b^{l_q} + B_2} = \frac{A_1 a^{m_p}}{B_1 b^{l_q}} \left(1 + \frac{A_2}{A_1 a^{m_p}}\right)\left(1 + \frac{B_2}{B_1 b^{l_q}}\right)^{-1}.$$

If x and y are real numbers with absolute values at most $\frac{1}{2}$ then

$$\max\left\{\frac{1+x}{1+y}, \frac{1+y}{1+x}\right\} \leqq 1 + 4\max\{|x|, |y|\}. \tag{10}$$

Certainly

$$\frac{|A_2|}{A_1 a^{m_p}} < \frac{2(b-1)(a-1)a^{m_{p+1}+1}}{(b-1)(a-1)a^{m_1}} \leqq 2a^{-m_1 + m_{p+1} + 1},$$

while, from (6), $m_1 - m_{p+1} \geqq \theta^s \geqq \theta \geqq c_1 \log\log n$ and thus for n sufficiently large $\frac{|A_2|}{A_1 a^{m_p}} < \frac{1}{2}$. In a similar manner we deduce that $\frac{|B_2|}{B_1 b^{l_q}} < \frac{1}{2}$. Therefore, putting $R = \frac{A_1 a^{m_p}}{B_1 b^{l_q}}$ and employing (10) we conclude that

$$1 \leqq \max\{R, R^{-1}\} \leqq 1 + 4 \max\left\{\frac{|A_2|}{A_1 a^{m_p}}, \frac{|B_2|}{B_1 b^{l_q}}\right\} \leqq 1 + 8 \max\{a^{-m_1+m_{p+1}+1}, b^{-l_1+l_{q+1}+1}\}.$$

Now, since $\log(1+x) < x$ for $x > 0$, we have

$$0 \leqq |\log R| \leqq 8ab \max\{a^{-m_1+m_{p+1}}, b^{-l_1+l_{q+1}}\}.$$

Therefore if $\log R \neq 0$ then (6) and (7) imply that

$$\log|\log R| < c_3 - c_4 \theta^s. \tag{11}$$

On the other hand we have

$$|\log R| = \left|\log\left(\frac{A_1}{B_1}\right) + m_p \log a - l_q \log b\right|$$

and so we may apply Lemma 1 to give a lower bound for $|\log R|$. We take $n=3, d=1$ and $\alpha_1, \alpha_2, \alpha_3$ to be $\frac{A_1}{B_1}$, a and b respectively. Note that m_p and l_q are at most $\frac{\log n}{\log 2}$ and that the height of $\frac{A_1}{B_1}$ is at most the maximum of A_1 and B_1. Therefore, by Lemma 1, if $\log R \neq 0$ then

$$|\log R| \geqq \exp(-c_5 \log(4\{\max A_1, B_1\}) \log\log n),$$

whence, from (8) and (9),

$$\log|\log R| \geqq -c_6(\max\{1, m_1 - m_p, l_1 - l_q\}) \log\log n,$$

which, from (6) and (7), yields

$$\log|\log R| \geqq -c_6 \theta^{s-1} \log\log n.$$

On comparing this estimate for $\log\log R$ with the one given by (11) we find that

$$c_4 \theta^s \leqq c_6 \theta^{s-1} \log\log n + c_3$$

hence

$$\theta \leqq c_7 \log\log n + c_8.$$

However, this contradicts (2) if c_1 is chosen to be larger than $c_7 + c_8$. Such a choice is possible since c_7 and c_8 have been determined independently of c_1. Thus we conclude that $\log R = 0$ and therefore

$$\log\left(\frac{A_1}{B_1}\right) + m_p \log a - l_q \log b = 0. \tag{12}$$

By Lemma 2 there exists a relation of the form

$$(13)\qquad x_1 \log\left(\frac{A_1}{B_1}\right) + x_2 \log a + x_3 \log b = 0,$$

with integer coefficients x_1, x_2, x_3, not all of which are zero, satisfying

$$\max\{|x_1|, |x_2|, |x_3|\} \leqq c_9 \log(\max\{A_1, B_1\}).$$

Recalling (6), (7), (8) and (9) we find that $|x_2| \leqq c_{10}\theta^{s-1} \leqq c_{10}\theta^{k-1}$; whence, from (2) and (3), $|x_2| < \frac{\log n}{4\log a}$ for n sufficiently large. Noting that $m_p \geqq m_1 - \theta^{s-1}$ and that $m_1 > \frac{\log n}{2\log a}$ we may employ (3) once again to verify that $m_p > \frac{\log n}{4\log a}$ whence $m_p > |x_2|$.

Now if $x_1 = 0$ it follows from (13) that $\frac{\log a}{\log b}$ is rational since x_1, x_2 and x_3 are not all zero. If $x_1 \neq 0$ then we find, on eliminating $\log\left(\frac{A_1}{B_1}\right)$ from equations (12) and (13), that

$$(m_p x_1 - x_2)\log a + (-l_q x_1 - x_3)\log b = 0.$$

Since m_p is larger than $|x_2|$ we again conclude that $\frac{\log a}{\log b}$ is rational. This completes the proof of Theorem 1.

4. The proof of Theorem 2

The proof of Theorem 2 is similar to the proof of Theorem 1. We first remark that λ_1 is real since it is strictly larger than all of its conjugates. By considering separately the sequences u_{2n} and u_{2n+1} we may assume that λ_1 is positive. Furthermore we may assume that u_n is non-negative. Thus, since $\lambda_1 > \max\{1, |\lambda_2|, \dots, |\lambda_k|\}$ and $P_1(x)$ is not identically zero we may assume that u_n is larger than a and that $P_1(n)$ is positive and we may write

$$u_n = P_1(n)\lambda_1^n + P_2(n)\lambda_2^n + \cdots + P_k(n)\lambda_k^n = a_1 a^{m_1} + \alpha\left(\frac{a^{m_1}-1}{a-1}\right) + a_2 a^{m_2} + \cdots + a_r a^{m_r};$$

where $0 < a_1 < a$ and $-\alpha \leqq a_i < a - \alpha$ with $a_i \neq 0$ for $i = 2, \dots, r$ and where in addition $m_1 > m_2 > \cdots > m_r \geqq 0$. We put

$$(14)\qquad \theta = c_1 \log n,$$

where c_1 is a positive number larger than 4 which is effectively computable in terms of $T = \{a, \lambda_1, \dots, \lambda_k, r$ and the coefficients of $P_1(x), \dots, P_k(x)\}$. We shall assume that n is at least c_2 where $c_2 > 4$ and where $c_2, c_3, \dots$ are positive numbers which are effectively computable in terms of T and which may be determined independently of the choice of c_1.

We may assume without loss of generality that $|\lambda_2| \geqq |\lambda_j|$ for $j=2,\ldots,k$ and therefore

$$|P_2(n)\lambda_2^n+\cdots+P_k(n)\lambda_k^n| \leqq c_3 n^r |\lambda_2|^n. \tag{15}$$

Since $\lambda_1 > |\lambda_2|$ and $P_1(n)$ is positive we have

$$\log\left(\frac{P_1(n)\lambda_1^n}{c_3 n^r |\lambda_2|^n}\right) > c_4 n \log a \tag{16}$$

and

$$\frac{\log u_n}{4\log a} > c_5 n, \tag{17}$$

for n sufficiently large. We put $c_6=\min\{c_4, c_5\}$. Finally let $B_1=(a-1)P_1(n)$ and denote the height of B_1 by V_1. A short calculation shows that V_1 is less than $c_7 n^{r^2}$ and thus

$$\log V_1 < c_8 \log n. \tag{18}$$

We consider now the intervals

$$\Theta_g=(\theta^{g-1}, \theta^g], \ldots, \Theta_k=(\theta^{k-1}, \theta^k],$$

where k satisfies the inequalities

$$\theta^k \leqq c_6 n < \theta^{k+1}, \tag{19}$$

and g satisfies the inequalities

$$\theta^{g-2} \leqq c_8 \log n < \theta^{g-1}. \tag{20}$$

If each interval Θ_i for $i=g,\ldots,k$ possesses at least one element of the form m_1-m_j the theorem holds since then

$$L_{\alpha,a}(u_n) \geqq r-1 \geqq k-g, \tag{21}$$

while from (14), (19) and (20) we have

$$k-g \geqq \frac{\log n}{\log c_1+\log\log n}-c_9. \tag{22}$$

Combining (21) and (22) and remarking that $L_{\alpha,a}(u_n)$ is always at least zero gives our result.

Thus we may assume that some Θ_s, with $g \leqq s \leqq k$, contains no term of the form m_1-m_j. We define p by the inequalities

$$m_1-m_p \leqq \theta^{s-1} \quad \text{and} \quad m_1-m_{p+1} > \theta^s, \tag{23}$$

again with the convention that $m_{r+1}=0$. As before we write $(a-1)u_n=A_1 a^{m_p}+A_2$, where A_1 and A_2 are integers satisfying

$$0 < A_1 < 2(a-1)a^{m_1-m_p+1}, \tag{24}$$

and

$$0 \leqq |A_2| < 2(a-1)a^{m_{p+1}+1}. \tag{25}$$

We also have $(a-1)u_n = B_1\lambda_1^n + B_2$, where $B_1 = (a-1)P_1(n)$ and

$$B_2 = (a-1)\left(P_2(n)\lambda_2^n + \cdots + P_k(n)\lambda_k^n\right).$$

From (25) and the observation that $A_1 a^{m_p} \geqq (a-1)a^{m_1}$ we find that

$$\frac{|A_2|}{A_1 a^{m_p}} < 2a^{-m_1+m_{p+1}+1},$$

and this is at most $\frac{1}{2}$ for n sufficiently large by (14) and (23). Furthermore, by (15),

$$\frac{|B_2|}{B_1\lambda_1^n} < \frac{c_{10} n^r |\lambda_2|^n}{\lambda_1^n},$$

which is also at most $\frac{1}{2}$ for n sufficiently large since $\lambda_1 > |\lambda_2|$. Putting $R = \frac{A_1 a^{m_p}}{B_1 \lambda_1^n}$ we may employ (10) as in the proof of Theorem 1 to conclude that

$$1 \leqq \max\{R, R^{-1}\} \leqq 1 + 4\max\left\{\frac{|A_2|}{A_1 a^{m_p}}, \frac{|B_2|}{B_1\lambda_1^n}\right\}$$

which from (15) is

$$\leqq 1 + 4\max\left\{2a^{-m_1+m_{p+1}+1}, \frac{c_3 n^r |\lambda_2|^n}{P_1(n)\lambda_1^n}\right\}.$$

Again, since $\log(1+x) < x$ for $x > 0$, we have

$$0 \leqq |\log R| \leqq 8a\max\left\{a^{-m_1+m_{p+1}}, \frac{c_3 n^r |\lambda_2|^n}{P_1(n)\lambda_1^n}\right\}.$$

Recalling (16) we see that if $\log R \neq 0$ then

$$\log|\log R| \leqq c_{11} + \max\{-(m_1 - m_{p+1})\log a, \ -c_4 n \log a\}$$

which by (23) is

$$\leqq c_{11} + \log a \max\{-\theta^s, \ -c_4 n\}.$$

Since $\theta^s \leqq \theta^k \leqq c_6 n \leqq c_4 n$ either $\log R = 0$ or

(26) $$\log|\log R| \leqq c_{11} - c_{12}\theta^s.$$

On the other hand, since λ_1 and a are positive real numbers,

$$|\log R| = \left|\log\left(\frac{A_1}{B_1}\right) + m_p \log a - n\log\lambda_1\right|,$$

where the principal values of the logarithms are taken. Thus by Lemma 1 if $\log R \neq 0$ then

$$|\log R| > \exp\left(-c_{13}(\max\{1, \log A_1, \log V_1\})\log n\right),$$

where V_1 is the height of B_1. Now by (18) and (20) $\log V_1 < c_8 \log n < \theta^{g-1} \leqq \theta^{s-1}$.

Furthermore from (23) and (24) we find that $\log A_1 < c_{14}\theta^{s-1}$. Therefore if $\log R \neq 0$ then

$$\log|\log R| > -c_{15}\theta^{s-1}\log n.$$

A comparison of this estimate with (26) reveals that

$$\theta < c_{16} + c_{17}\log n,$$

and this contradicts (14) if c_1 is taken to be larger than $c_{16}+c_{17}$. Thus we may conclude that

$$\log\left(\frac{A_1}{B_1}\right) + m_p \log a - n\log\lambda_1 = 0. \tag{27}$$

Since λ_1 and a are both positive real numbers $\frac{A_1}{B_1}$ is also a positive real number. Thus we may apply Lemma 2 to obtain the relation

$$x_1\log\left(\frac{A_1}{B_1}\right) + x_2\log a + x_3\log\lambda_1 = 0, \tag{28}$$

where x_1, x_2 and x_3 are integers, not all of which are zero, satisfying

$$\max\{|x_1|, |x_2|, |x_3|\} < c_{18}\theta^{s-1}.$$

Recall from (23) that $m_p \geqq m_1 - \theta^{s-1}$ and observe that $m_1 \geqq \frac{\log u_n}{2\log a}$ and, by (17) and (19), that $\theta^s \leqq \theta^k \leqq c_6 n \leqq \frac{\log u_n}{4\log a}$. Therefore $m_p \geqq \frac{\log u_n}{4\log a} \geqq \theta^s$ whence

$$m_p > c_{18}\theta^{s-1} > |x_2|$$

for n sufficiently large. Eliminating $\log\left(\frac{A_1}{B_1}\right)$ from equations (27) and (28) as in the proof of Theorem 1 we conclude that $\frac{\log\lambda_1}{\log a}$ is rational. This completes the proof of Theorem 2.

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

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