Pure Mathematics Linear Algebra Qualifying Examination
University of Waterloo
September 27, 2022

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let $T\colon V \to V$ be a unitary linear transformation on a finite dimensional inner product space V. Prove that there is a unitary linear transformation $U\colon V \to V$ such that $U \circ U = T$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. Let T_1 and T_2 be linear transformations from a finite-dimensional vector space V to itself, and assume that both T_1 and T_2 are diagonalizable. That is, assume that there are bases B_1 and B_2 such that the matrices $[T_i]_{B_i}$ are diagonal. Prove that $T_1T_2 = T_2T_1$ if and only if there is a basis B such that the matrices $[T_i]_B$ are both diagonal.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let V be the span of the functions $\{e^x, xe^x, x^2e^x, e^{2x}\}$. (You may assume that the dimension of V is four.) Define $T\colon V \to V$ by $T(f) = f'$, where f' denotes the derivative of f with respect to x. Find a Jordan canonical basis for T.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

4. Let V be a real vector space, and let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V. Say that V admits a pairing $\langle \cdot, \cdot \rangle$ which satisfies for all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$ and $c \in \mathbb{R}$:

   - $\langle \mathbf{v} + \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{x} \rangle + \langle \mathbf{w}, \mathbf{x} \rangle$
   - $\langle c\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle$
   - $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$

   Note that this pairing may not satisfy the positive definite property.

   Assume that there is some $\mathbf{v} \in V$ such that $\langle \mathbf{v}, \mathbf{v} \rangle > 0$. Prove that there is a basis $B' = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ such that for each i:

$$\langle \mathbf{w}_i, \mathbf{w}_i \rangle > 0$$

   However, show by example that V need not be an inner product space with this pairing.

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