On divisors of sums of integers · II

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1. Introduction

Throughout this article, $c_0, c_1, c_2, \ldots$ will denote effectively computable positive absolute constants. Denote the cardinality of a set X by $|X|$ and for any integer n let $P(n)$ denote the greatest prime factor of n with the convention that $P(0)=P(\pm 1)=1$. Let N be a positive integer and let A and B be non-empty subsets of $\{1, \ldots, N\}$. In [2] Balog and Sárközy proved, by means of the large sieve, that if $N>N_0$, and $|A|\,|B|>100N(\log N)^2$ then there exist $a \in A$ and $b \in B$ such that

$$P(a+b)>\frac{(|A|\,|B|)^{\frac{1}{2}}}{16\log N}. \tag{1}$$

Thus, in particular, if $|A| \gg N$ and $|B| \gg N$ then there exist $a \in A$ and $b \in B$ with

$$P(a+b) \gg \frac{N}{\log N}. \tag{2}$$

In part I of this paper [9], we obtained estimates for the greatest prime factor of sums of integers taken from k sets. In this paper we shall return to the case $k=2$. Our aim is to improve upon (1) when $|A|$ and $|B|$ are close to N. For example, we shall show that the right hand side of (2) may be replaced by N, which of course is best possible. Further we shall show that there exist many pairs (a, b) with a in A and b in B for which $P(a+b)$ is large.

Put

$$R=\frac{3N}{(|A|\,|B|)^{\frac{1}{2}}}.$$

Theorem. *Let N be a positive integer, let A and B be subsets of $\{1, \ldots, N\}$ and let ε be a positive real number. There exist effectively computable positive absolute constants c_1, c_2, c_3 and c_4 and a positive number N_1 which is effectively computable in terms of ε such that if $N>N_1$ and*

$$(|A|\,|B|)^{\frac{1}{2}}>N^{\frac{5}{6}+\varepsilon}, \tag{3}$$

*) The research of the second author was supported in part by Grant A3528 from the Natural Sciences and Engineering Research Council of Canada.

then there exist at least $\frac{c_1|A|\,|B|}{\log N}$ *pairs* (a, b) *with* a *in* A *and* b *in* B *for which*

$$\frac{2c_2(|A|\,|B|)^{\frac{1}{2}}}{\log R \log\log R} \geqq P(a+b) > \frac{c_2(|A|\,|B|)^{\frac{1}{2}}}{\log R \log\log R}, \tag{4}$$

and there exist at least $\frac{c_3|A|\,|B|}{\log N}$ *pairs* (a_1, b_1) *with* a_1 *in* A *and* b_1 *in* B *for which*

$$\frac{2c_4(|A|\,|B|)^{\frac{1}{2}}}{\log R \log\log R} \geqq P(a_1-b_1) > \frac{c_4(|A|\,|B|)^{\frac{1}{2}}}{\log R \log\log R}. \tag{5}$$

We remark that the estimates for the number of pairs (a, b) satisfying (4) and (5) can not be substantially improved. For example if $A = B = \{1, \ldots, N\}$ then the number of pairs satisfying (4) is at most $c_1^* \frac{N^2}{\log N} = c_1^* \frac{|A|\,|B|}{\log N}$, where c_1^* is a positive real number which is effectively computable in terms of c_2. Further, let T be a positive real number with

$$\frac{c_2(|A|\,|B|)^{\frac{1}{2}}}{\log R \log\log R} > T > N^{\frac{5}{6}+\varepsilon}.$$

On applying the above theorem to subsets of A and B of the appropriate size we find that there exist a in A and b in B with $3T > P(a+b) > T$, provided that N is larger than a number which is effectively computable in terms of ε.

In particular, if (3) holds then for N sufficiently large there exist a in A and b in B such that

$$P(a+b) > \frac{c_2(|A|\,|B|)^{\frac{1}{2}}}{\log R \log\log R}, \tag{6}$$

and there exist a_1 in A and b_1 in B with $a_1 \neq b_1$ such that

$$P(a_1-b_1) > \frac{c_4(|A|\,|B|)^{\frac{1}{2}}}{\log R \log\log R}. \tag{7}$$

Thus if $|A| \gg N$ and $|B| \gg N$ then there exist a in A and b in B such that

$$P(a+b) \gg N,$$

and a_1 in A, b_1 in B with $a_1 \neq b_1$ such that

$$P(a_1-b_1) \gg N.$$

Notice that (6) yields an improvement on (1) provided that

$$|A|\,|B| > N^2 \exp\left(-c_0\left(\frac{\log N}{\log\log N}\right)\right).$$

Furthermore, we remark that if A and B consist of all multiples of a positive integer t with $t \leqq N^{\frac{1}{2}}$ then for all a in A and b in B,

$$P(a \pm b) \leqq \max\left(P(t), 2\left[\frac{N}{t}\right]\right) \leqq 2\left[\frac{N}{t}\right] \leqq 2(|A|\,|B|)^{\frac{1}{2}},$$

hence (6) and (7) are nearly best possible even when $|A|\,|B| = o(N^2)$.

Perhaps for any $\varepsilon > 0$ there exist $N_0(\varepsilon)$ and $K = K(\varepsilon)$ such that if $N > N_0(\varepsilon)$ and $|A|\,|B| > KN$ then there exist $a \in A$ and $b \in B$ such that

$$P(a+b) > (2-\varepsilon)\,(|A|\,|B|)^{\frac{1}{2}}, \tag{8}$$

and $a_1 \in A$, $b_1 \in B$ with $a_1 \neq b_1$ such that

$$P(a_1 - b_1) > (1-\varepsilon)\,(|A|\,|B|)^{\frac{1}{2}}. \tag{9}$$

The following simple construction shows that the hypothesis $|A|\,|B| > KN$ is necessary in the above conjecture. Let γ be a real number with $0 < \gamma < \frac{1}{2}$ and let $n_1, \dots, n_k$ be those positive integers n_i with $2 \leqq n_i \leqq N$ and $P(n_i) \leqq N^\gamma$. Put $A = \{1\}$ and $B = \{n_1 - 1, n_2 - 1, \dots, n_k - 2\}$. By Lemma 3. 20 and Lemma 4. 7 of [7] there exists a positive number $c(\gamma)$ such that $|B| > c(\gamma)\,N$ for N sufficiently large. We then have

$$|A|\,|B| > c(\gamma)\,N,$$

while $P(a+b) \leqq N^\gamma$ for all $a \in A$ and $b \in B$.

§ 2. Preliminary lemmas

For any real number x let $[x]$ denote the greatest integer less than or equal to x, let $\{x\} = x - [x]$ denote the fractional part of x and let

$$\|x\| = \min(\{x\}, 1 - \{x\})$$

denote the distance from x to the nearest integer. Further denote $e^{2\pi i x}$ by $e(x)$.

Lemma 1. *Let X and Y be positive integers with $X < Y$. Then for any real number α we have*

$$\left|\sum_{X < n \leqq Y} e(n\alpha)\right| \leqq \min\left(Y - X, \frac{1}{2\|\alpha\|}\right).$$

Proof. See [8], p. 189.

Lemma 2. *Let V be a positive integer. Then for any real number α we have*

$$\left|\sum_{n=0}^{V-1} e(n\alpha) - V\right| \leqq 4\,V^2\,|\alpha|.$$

Proof. See [1], Lemma 2.

For any positive integer n denote the number of integers less than or equal to n and coprime with n by $\phi(n)$. ϕ is Euler's phi function.

Lemma 3. *There exists an effectively computable positive real number c_5 such that*

$$\phi(n) > c_5 \frac{n}{\log\log n},$$

for $n \geqq 3$.

Proof. See [8], p. 24.

For any positive integer n, denote the number of positive integers which divide n by $\tau(n)$.

Lemma 4. *Let q be a positive integer and let u and v be real numbers with $v > 0$. Then*

$$\left| \sum_{\substack{u<k\leqq u+v \\ (k,q)=1}} 1 - v\frac{\phi(q)}{q} \right| \leqq \tau(q). \tag{10}$$

Proof.

$$\left| \sum_{\substack{u<k\leqq u+v \\ (k,q)=1}} 1 - v\frac{\phi(q)}{q} \right| = \left| \sum_{u<k\leqq u+v} \sum_{d|(k,q)} \mu(d) - v\frac{\phi(q)}{q} \right|$$

$$= \left| \sum_{d|q} \mu(d) \sum_{\substack{u<k\leqq u+v \\ d|k}} 1 - v \sum_{d|q} \frac{\mu(d)}{d} \right|$$

$$= \left| \sum_{d|q} \mu(d) \left(\sum_{\substack{u<k\leqq u+v \\ d|k}} 1 - \frac{v}{d} \right) \right| \leqq \sum_{d|q} 1 = \tau(q).$$

Lemma 5. *There exists an effectively computable positive real number c_6 such that for any integers a and b with $b \geqq 2$,*

$$\frac{b}{\phi(b)} \sum_{\substack{1\leqq n\leqq b \\ (n+a,b)=1}} \frac{1}{n} < c_6 \log b.$$

Proof. Since the result plainly holds for $b=2$ we may assume that $b \geqq 3$. First we notice that the result holds with $a=0$. We have

$$\sum_{\substack{n=1 \\ (n,b)=1}}^{b} \frac{1}{n} < \prod_{\substack{p\leqq b \\ p\nmid b}} \left(1-\frac{1}{p}\right)^{-1}$$

hence

$$\frac{b}{\phi(b)} \sum_{\substack{n=1 \\ (n,b)=1}}^{b} \frac{1}{n} < \prod_{p\leqq b} \left(1-\frac{1}{p}\right)^{-1}$$

and, by Theorem 429 of [4],

$$\frac{b}{\phi(b)} \sum_{\substack{n=1 \\ (n,b)=1}}^{n} \frac{1}{n} < c_7 \log b. \tag{11}$$

Next put

$$t = \max_{u,v\in Z^+} \left| \sum_{\substack{u<k\leqq u+v \\ (k,b)=1}} 1 - \sum_{\substack{0<k\leqq v \\ (k,b)=1}} 1 \right|,$$

where $\mathbb{Z}^+$ denotes the set of positive integers. We have

$$\sum_{\substack{n=1 \\ (n+a,b)=1}}^{b} \frac{1}{n} \leqq \sum_{\substack{n=1 \\ (n,b)=1}}^{b} \frac{1}{n} + \sum_{n=1}^{t} \frac{1}{n},$$

since the j-th term in the sum on the left above is at most $\frac{1}{j}$ for $1 \leqq j \leqq t$ and is at most the $(j-t)$-th term in $\sum_{\substack{n=1 \\ (n,b)=1}}^{b} \frac{1}{n}$ for $t < j \leqq \phi(b)$. Therefore, by (11),

$$\frac{b}{\phi(b)} \sum_{\substack{n=1 \\ (n+a,b)=1}}^{b} \frac{1}{n} < c_7 \log b + \frac{b}{\phi(b)} (1 + \log t)$$

which, by Lemma 3, is

$$< c_7 \log b + c_8 \log\log b (1 + \log t).$$

It follows from Lemma 4 that $t \leqq 2\tau(b)$. Further, by Theorem 317 of [4],

$$\tau(b) \leqq b^{\frac{c_9}{\log\log b}},$$

and thus $(1 + \log t) \leqq c_{10} \frac{\log b}{\log\log b}$. Therefore

$$\frac{b}{\phi(b)} \sum_{\substack{n=1 \\ (n+a,b)=1}}^{b} \frac{1}{n} < c_{11} \log b,$$

as required.

Lemma 6. *Let h, a and q be integers with $a > 0$, $q > 1$ and $(a, q) = 1$. Let $\rho(n)$ be a real valued function defined for those integers n with $h \leqq n < h + q$ and $(n, q) = 1$. Put*

$$\lambda = \max_{\substack{h \leqq n < h+q \\ (n,q)=1}} \rho(n) - \min_{\substack{h \leqq n < h+q \\ (n,q)=1}} \rho(n),$$

and

$$\psi(n) = \frac{1}{q} (an + \rho(n)).$$

There is an effectively computable positive absolute constant c_{12} such that if $\lambda \leqq 1$ and if E is a real number satisfying $2 \leqq E \leqq q$ then

$$\sum_{\substack{h \leqq n < h+q \\ (n,q)=1}} \min\left(E, \frac{1}{\|\psi(n)\|}\right) < c_{12} \phi(q) \log E.$$

Proof. Put $r = \left[\min_{\substack{h \leqq n < h+q \\ (n,q)=1}} \rho(n) \right]$ and $\rho_1(n) = \rho(n) - r$. Note

$$0 \leqq \rho_1(n) \leqq \lambda + 1 \leqq 2.$$

We have $\psi(n)=\frac{1}{q}((an+r)+\rho_1(n))$ and so

$$\frac{an+r}{q}\leqq\psi(n)\leqq\frac{an+r+2}{q},$$

hence

$$\frac{1}{\|\psi(n)\|}\leqq\max\left(\frac{1}{\left\|\frac{an+r}{q}\right\|},\frac{1}{\left\|\frac{an+r+1}{q}\right\|},\frac{1}{\left\|\frac{an+r+2}{q}\right\|}\right),$$

subject to the convention that $a\leqq\max\left(\frac{1}{0},b\right)$ and $\frac{1}{0}\leqq\max\left(\frac{1}{0},a\right)$ for all real numbers a and b. Thus

$$\sum_{\substack{h\leqq n<h+q\\(n,q)=1}}\min\left(E,\frac{1}{\|\psi(n)\|}\right)\leqq\sum_{\substack{h\leqq n<h+q\\(n,q)=1}}\sum_{i=0}^{2}\min\left(E,\left\|\frac{an+r+i}{q}\right\|^{-1}\right)$$

$$\leqq 3\max_{j\in Z}\sum_{\substack{h\leqq n<h+q\\(n,q)=1}}\min\left(E,\left\|\frac{an+j}{q}\right\|^{-1}\right)$$

which, since $(a,q)=1$, is

$$\leqq 3\max_{j\in Z}\sum_{\substack{-j\leqq v<-j+q\\(v,q)=1}}\min\left(E,\left\|\frac{v+j}{q}\right\|^{-1}\right)$$

$$\leqq 3\max_{j\in Z}\left(E+\sum_{\substack{1\leqq t\leqq q-1\\(t-j,q)=1}}\min\left(E,\left\|\frac{t}{q}\right\|^{-1}\right)\right),$$

and since $\left\|\frac{t}{q}\right\|^{-1}\leqq\max\left(\frac{q}{t},\frac{q}{q-t}\right)$ for $1\leqq t\leqq q-1$, we have, on putting $q-t=x$, that the above is

$$\leqq 3\max_{j\in Z}\left(E+\sum_{\substack{1\leqq t\leqq q-1\\(t-j,q)=1}}\min\left(E,\frac{q}{t}\right)+\sum_{\substack{1\leqq x\leqq q-1\\(x+j,q)=1}}\min\left(E,\frac{q}{x}\right)\right)$$

$$\leqq 6\max_{j\in Z}\left(E+\sum_{\substack{1\leqq t\leqq q-1\\(t-j,q)=1}}\min\left(E,\frac{q}{t}\right)\right), \tag{12}$$

$$\leqq 6E+6q\max_{j\in Z}\left(\sum_{\substack{1\leqq t\leqq q-1\\(t-j,q)=1}}\frac{1}{t}\right),$$

which, by Lemma 5 is

$$\leqq 6E+6c_6\phi(q)\log q. \tag{13}$$

If $q\geqq E\geqq q^{\frac{1}{3}}$ then it follows from Lemma 3 and (13) that

$$\sum_{\substack{h\leqq n<h+q\\(n,q)=1}}\min\left(E,\frac{1}{\|\psi(n)\|}\right)\leqq c_{13}\phi(q)\log E. \tag{14}$$

On the other hand if $2 \leqq E \leqq q^{\frac{1}{3}}$ then the right hand side of inequality (12) is

$$\leqq 6 \max_{j \in Z} \left(E + \sum_{\substack{1 \leqq t < \frac{q}{E} \\ (t-j,q)=1}} E + \sum_{m=1}^{[E]} \sum_{\substack{\frac{mq}{E} \leqq t < (m+1)\frac{q}{E} \\ (t-j,q)=1}} \frac{q}{t} \right)$$

which, by Lemma 4, is

$$\leqq 6 \left(E + E\left(\frac{\phi(q)}{E} + \tau(q) \right) + \sum_{m=1}^{[E]} \frac{E}{m} \left(\frac{\phi(q)}{E} + \tau(q) \right) \right).$$

Since $\tau(q) \leqq 2q^{\frac{1}{2}}$ for all positive integers q, and $E < q^{\frac{1}{3}}$ it follows from Lemma 3 that

$$\frac{\phi(q)}{E} + \tau(q) < c_{14} \frac{\phi(q)}{E}.$$

Therefore

$$\sum_{\substack{h \leqq n < h+q \\ (n,q)=1}} \min\left(E, \frac{1}{\|\psi(n)\|} \right) \leqq 6 \left(E + c_{14}\phi(q) + c_{14}\phi(q) \sum_{m=1}^{[E]} \frac{1}{m} \right)$$

which by Lemma 3 is

$$\leqq c_{15}\phi(q) + c_{16}\phi(q) \log E \leqq c_{17}\phi(q) \log E. \tag{15}$$

Our result follows from (14) and (15).

We shall also require the Brun-Titchmarsh theorem, a result of Heath-Brown and Iwaniec and a refinement, due to Vaughan, of Vinogradov's fundamental lemma on exponential sums.

Let x be a positive real number and let l and k be positive integers. As usual we denote the number of primes less than or equal to x by $\pi(x)$ and the number of primes less than or equal to x and congruent to l modulo k by $\pi(x, k, l)$.

Lemma 7 (Brun-Titchmarsh Theorem). *Let x and y be positive real numbers and let k and l be relatively prime positive integers with $y > k$. Then*

$$\pi(x+y, k, l) - \pi(x, k, l) < \frac{2y}{\phi(k) \log\left(\frac{y}{k}\right)}.$$

Proof. See Theorem 2 of [6].

Lemma 8. *Given $\varepsilon > 0$ there exist positive real numbers $C_0 = C_0(\varepsilon)$ and $x_0 = x_0(\varepsilon)$, which are effectively computable in terms of ε, such that if $y \geqq x^{\frac{11}{20}+\varepsilon}$ and $x > x_0$ then*

$$\pi(x+y) - \pi(x) > C_0 \frac{y}{\log y}.$$

Proof. See [5].

We remark that for our purposes we require Lemma 8 only for the range $y \geqq x^{\frac{3}{5}}$.

Lemma 9. *If α is a real number and a, q and N are positive integers with $(a, q)=1$, $q \leqq N$ and $\left|\alpha-\frac{a}{q}\right| \leqq q^{-2}$ then*

$$\left|\sum_{p \leqq N} e(p\alpha)\right| < c_{18}(\log N)^4 \left(Nq^{-\frac{1}{2}}+N^{\frac{4}{5}}+N^{\frac{1}{2}}q^{\frac{1}{2}}\right),$$

where c_{18} is an effectively computable positive absolute constant; the summation above is over primes p with $p \leqq N$.

Proof. This follows from Theorem 3.1 of [10] by partial summation.

§ 3. Further preliminaries

In order to prove our main theorem we shall employ the Hardy-Littlewood method much as in [2]. In fact, apart from the values of the parameters, we shall start out from the same integral. However the integral must be estimated in a much more elaborate way. In particular, the treatment of the „major arcs" requires new ideas.

Let ε be a positive real number less than $\frac{1}{6}$ and let $N_1, N_2, \ldots$ denote positive numbers which are effectively computable in terms of ε. Put

$$y=\omega R \log R \log\log R$$

where ω is a positive real number larger than 400 which is effectively computable. Since $R \geqq 3$, $y \geqq (3 \log 3 \log\log 3)\,\omega \geqq \frac{\omega}{4}$, hence

$$y > 100. \tag{16}$$

Further, if (3) holds and $N > N_5$ *then*

$$y < 3\,\omega N^{\frac{1}{6}-\varepsilon} \log N^{\frac{1}{6}} \log\log N^{\frac{1}{6}} < \frac{N^{\frac{1}{6}-\frac{\varepsilon}{2}}}{2(\log N)^5} \tag{17}$$

and so

$$y < N^{\frac{1}{6}}. \tag{18}$$

We shall first establish (4). To do so it suffices to show that there exist at least $c_1 \frac{|A|\,|B|}{\log N}$ pairs (a, b) with a in A and b in B for which

$$\frac{4N}{y} \geqq P(a+b) > \frac{2N}{y}. \tag{19}$$

To this end we introduce the following notation. Put

$$Q=\frac{N}{y^3(\log N)^{10}}, \quad \delta=\frac{y}{8N}, \quad U=\left[\frac{N}{y^2}\right]$$

and, for each positive integer n,

$$d_n=\begin{cases} 1 & \text{if } n=mp \text{ with } 1 \leqq m \leqq y \text{ and } p \text{ a prime such that } \frac{2N}{y} < p \leqq \frac{4N}{y}, \\ 0 & \text{otherwise.}\end{cases}$$

Next put

$$S(\alpha)=\sum_{n=1}^{4N} d_n e(n\alpha),$$

$$S=S(0)=\sum_{n=1}^{4N} d_n,$$

$$U(\alpha)=\sum_{n=0}^{U-1} e(n\alpha),$$

and, since $d_n=0$ if $n<1$ or $n>4N$, write

$$S(\alpha)\,U(\alpha)=\sum_{n=1}^{4N+U-1} v_n e(n\alpha) \quad \text{where} \quad v_n=\sum_{j=n-U+1}^{n} d_j.$$

Further, put

$$F(\alpha)=\sum_{a\in A} e(a\alpha), \quad G(\alpha)=\sum_{b\in B} e(b\alpha)$$

and

$$H(\alpha)=F(\alpha)\,G(\alpha)=\sum_{a\in A,\, b\in B} e\big((a+b)\,\alpha\big)=\sum_{n=1}^{2N} h_n e(n\alpha)$$

where

$$h_n=\sum_{\substack{a+b=n\\ a\in A,\, b\in B}} 1.$$

Finally, define J by

$$J=\int_0^1 F(\alpha)\,G(\alpha)\,S(-\alpha)\,d\alpha.$$

Observe that

$$J=\int_0^1 H(\alpha)\,S(-\alpha)\,d\alpha=\int_0^1 \sum_{n=1}^{2N}\sum_{m=1}^{4N} h_n d_m e\big((n-m)\,\alpha\big)\,d\alpha$$
$$=\sum_{n=1}^{2N} h_n d_n.$$

Also note that $d_n>0$ implies that $\frac{4N}{y}\geqq P(n)>\frac{2N}{y}$ while $h_n>0$ implies that n can be expressed as $a+b$ for some $a\in A$ and $b\in B$. Thus, to establish (19) and hence also (4), it suffices to show that

(20) $$J>c_1\frac{|A|\,|B|}{\log N}.$$

In order to prove (20) we shall first establish estimates for S, $S(\alpha)$ and v_n.

It follows from (18) that, for $N>N_5$, $y<\frac{2N}{y}$ and therefore

(21) $$S(\alpha)=\sum_{m\leqq y}\sum_{\frac{2N}{y}<p\leqq\frac{4N}{y}} e(mp\alpha).$$

Lemma 10. For $N > N_5$ we have

$$S < 10 \frac{N}{\log N}.$$

Proof. By (21), for $N > N_5$,

$$S = \sum_{n=1}^{4N} d_n = \left(\sum_{1 \leqq m \leqq y} 1 \right) \left(\sum_{\frac{2N}{y} < p \leqq \frac{4N}{y}} 1 \right) \leqq y \pi \left(\frac{4N}{y} \right),$$

which, by Lemma 7 with $k = 1$, is

$$\leqq \frac{8N}{\log\left(\frac{4N}{y}\right)} < \frac{8N}{\log(N/N^{\frac{1}{6}})} < 10 \frac{N}{\log N}.$$

Lemma 11. *If* $N > N_6$ *then*

$$|S(\alpha)| < c_{19} \frac{N \log y \log\log y}{y \log N},$$

for $\delta < \alpha < 1 - \delta$.

Proof. Let T_1 denote the set of those α in the interval $(\delta, 1-\delta)$ for which for all integers n with $1 \leqq n \leqq y$ there exist positive integers r_n and s_n with $(r_n, s_n) = 1$,

$$\left| n\alpha - \frac{r_n}{s_n} \right| < \frac{1}{s_n^2}, \tag{22}$$

and

$$y^2 (\log N)^{10} \leqq s_n \leqq Q. \tag{23}$$

Put $T' = (\delta, 1-\delta) - T_1$, so that T' consists of the real numbers α in $(\delta, 1-\delta)$ which are not in T_1. If $\alpha \in T'$, then for some integer n^* with $1 \leqq n^* \leqq y$ there exist no coprime positive integers r_{n^*}, s_{n^*} satisfying (22) and (23) with n^* in place of n. By Dirichlet's theorem there exist integers u and v with

$$\left| n^* \alpha - \frac{u}{v} \right| < \frac{1}{vQ}, \tag{24}$$

$0 \leqq u$, $0 < v \leqq Q$ and $(u, v) = 1$. Note that

$$\left| n^* \alpha - \frac{u}{v} \right| < \frac{1}{v^2},$$

and therefore that $v < y^2 (\log N)^{10}$. It follows directly from (24) that

$$\left| \alpha - \frac{u}{n^* v} \right| < \frac{1}{n^* v Q}$$

hence, on writing $\frac{u}{n^* v}$ in the form $\frac{a}{b}$ with a and b coprime, $a \geqq 0$ and $b > 0$ we see that

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{bQ} \tag{25}$$

with

$$b \leqq n^* v \leqq y^3 (\log N)^{10}. \tag{26}$$

To each α in T' we shall associate a pair of coprime integers a and b with $a \geqq 0$ and $b > 0$ satisfying (25) and (26) and we shall put

$$\beta = \alpha - \frac{a}{b}.$$

Let us define subsets T_2, T_3 and T_4 of T' in the following way:

$$T_2 = \left\{\alpha \in T' \mid 1 \leqq b \leqq y,\ |\beta| \leqq \frac{y}{8bN}\right\}.$$

$$T_3 = \left\{\alpha \in T' \mid 1 \leqq b \leqq y,\ |\beta| > \frac{y}{8bN}\right\}.$$

$$T_4 = \{\alpha \in T' \mid y < b\}.$$

Since $(\delta, 1-\delta) = T_1 \cup T_2 \cup T_3 \cup T_4$ it suffices to show that

$$\max_{\alpha \in T_i} |S(\alpha)| < c_{19} \frac{N \log y \log\log y}{y \log N}, \tag{27}$$

for $i = 1, 2, 3, 4$ and for $N > N_6$.

We shall first establish (27) for $i = 1$. Accordingly assume that $\alpha \in T_1$. By (21), for $N > N_5$,

$$|S(\alpha)| = \left|\sum_{n \leqq y} \sum_{\frac{2N}{y} < p \leqq \frac{4N}{y}} e(np\alpha)\right| \tag{28}$$

$$\leqq \sum_{n \leqq y} \left(\left|\sum_{p \leqq \frac{4N}{y}} e(p(n\alpha))\right| + \left|\sum_{p \leqq \frac{2N}{y}} e(p(n\alpha))\right|\right),$$

which by Lemma 9, (22) and (23), is

$$\leqq \sum_{n \leqq y} c_{20} \left(\log \frac{4N}{y}\right)^4 \left(\frac{4N}{y}\left(y^2 (\log N)^{10}\right)^{-\frac{1}{2}} + \left(\frac{4N}{y}\right)^{\frac{4}{5}} + \left(\frac{4N}{y}\right)^{\frac{1}{2}} Q^{\frac{1}{2}}\right)$$

$$\leqq c_{21} y (\log N)^4 \left(\frac{6N}{y^2 (\log N)^5} + \left(\frac{4N}{y}\right)^{\frac{4}{5}}\right)$$

and, by (17),

$$\leqq c_{22} \frac{N}{y \log N}, \tag{29}$$

for $N > N_7$.

Now assume that $\alpha \in T_2$. Notice that we may assume that $b>1$ since if $b=1$ then $|\beta| \leqq \frac{y}{8N} = \delta$ and consequently α is not in $(\delta, 1-\delta)$. Further since $b \neq 1$ we may assume that $a \neq 0$. We have, from (28),

$$|S(\alpha)| \leqq \sum_{\frac{2N}{y} < p \leqq \frac{4N}{y}} \left| \sum_{n \leqq y} e(np\alpha) \right|$$

which, by Lemma 1, is

$$\leqq \sum_{\frac{2N}{y} < p \leqq \frac{4N}{y}} \min\left(y, \frac{1}{2\|p\alpha\|}\right).$$

It follows from (18) and (26) that if p is a prime with $p > \frac{2N}{y}$ and $N > N_8$ then $(p, b) = 1$. Thus for $N > N_9$

$$|S(\alpha)| \leqq \sum_{\substack{p \leqq \frac{4N}{y} \\ (p,b)=1}} \min\left(y, \frac{1}{2\|p\alpha\|}\right). \tag{30}$$

Notice that

$$\|p\alpha\| = \left\| p\left(\frac{a}{b} + \beta\right) \right\| \geqq \left\| \frac{ap}{b} \right\| - p|\beta| \geqq \left\| \frac{ap}{b} \right\| - \left(\frac{4N}{y}\right)\left(\frac{y}{8bN}\right)$$
$$= \left\| \frac{ap}{b} \right\| - \frac{1}{2b} \geqq \frac{1}{2} \left\| \frac{ap}{b} \right\|,$$

since $b>1$ and $(ap, b) = 1$. Thus, from (30), for $N > N_{10}$

$$|S(\alpha)| \leqq \sum_{\substack{p \leqq \frac{4N}{y} \\ (p,b)=1}} \min\left(y, \frac{1}{\left\| \frac{ap}{b} \right\|}\right)$$

$$\leqq \sum_{\substack{0 \leqq h < b \\ (h,b)=1}} \left(\sum_{\substack{p \leqq \frac{4N}{y} \\ ap \equiv h \pmod b}} \frac{1}{\left\| \frac{h}{b} \right\|} \right)$$

$$\leqq \left(\max_{\substack{0<l<b \\ (l,b)=1}} \pi\left(\frac{4N}{y}, b, l\right) \right) \sum_{\substack{0 \leqq h < b \\ (h,b)=1}} \frac{1}{\left\| \frac{h}{b} \right\|}$$

which, by Lemma 7, (18) and (26), is

$$\leqq \frac{8N}{y\phi(b) \log\left(\frac{4N}{by}\right)} \cdot 2 \sum_{\substack{1 \leqq h \leqq \frac{b}{2} \\ (h,b)=1}} \frac{b}{h}$$

and, by Lemma 5, this is

$$\leqq c_{23} \frac{N}{y \log N} \log b \leqq c_{23} \frac{N}{y \log N} \log y, \tag{31}$$

as required.

We shall assume next that $\alpha \in T_3$ hence

(32) $$\frac{y}{8bN} < |\beta| < \frac{1}{bQ}.$$

Put

(33) $$L = \frac{1}{2b|\beta|}.$$

It follows from (32) that

$$\frac{Q}{2} < L < \frac{4N}{y}.$$

Then, from (30), for $N > N_9$,

$$|S(\alpha)| \leqq \sum_{\substack{p \leqq \frac{4N}{y} \\ (p,b)=1}} \min\left(y, \frac{1}{2\|p\alpha\|}\right)$$
$$\leqq \sum_{j=1}^{\left[\frac{4N}{Ly}\right]+1} \sum_{\substack{(j-1)L < p \leqq jL \\ (p,b)=1}} \min\left(y, \frac{1}{2\|p\alpha\|}\right)$$
$$= \sum_{j=1}^{\left[\frac{4N}{Ly}\right]+1} \sum_{k=1}^{2y} \sum_{\substack{(j-1)L < p \leqq jL \\ (p,b)=1 \\ \frac{k-1}{2y} \leqq \{p\alpha\} < \frac{k}{2y}}} \min\left(y, \frac{1}{2\|p\alpha\|}\right).$$

Since $\frac{k-1}{2y} \leqq \{p\alpha\} < \frac{k}{2y}$ implies that

$$\frac{1}{\|p\alpha\|} \leqq \frac{1}{\left\|\frac{k-1}{2y}\right\|} + \frac{1}{\left\|\frac{k}{2y}\right\|},$$

where as before we write $a \leqq \frac{1}{0} + b$ and $\frac{1}{0} \leqq \frac{1}{0} + a$ for all real numbers a and b, we have

(34) $$|S(\alpha)| \leqq \sum_{j=1}^{\left[\frac{4N}{Ly}\right]+1} \sum_{k=1}^{2y} \left(\min\left(y, \frac{1}{2\left\|\frac{k-1}{2y}\right\|}\right) + \min\left(y, \frac{1}{2\left\|\frac{k}{2y}\right\|}\right)\right) \sum_{\substack{(j-1)L < p \leqq jL \\ (p,b)=1 \\ \frac{k-1}{2y} \leqq \{p\alpha\} < \frac{k}{2y}}} 1.$$

If p and p_0 are primes with $(j-1)L < p \leqq jL$, $\frac{k-1}{2y} \leqq \{p\alpha\} < \frac{k}{2y}$ and $(j-1)L < p_0 \leqq jL$, $\frac{k-1}{2y} \leqq \{p_0\alpha\} < \frac{k}{2y}$ then

$$\frac{1}{2y} > \|(p-p_0)\alpha\| = \left\|(p-p_0)\left(\frac{a}{b}+\beta\right)\right\| \geqq \left\|(p-p_0)\frac{a}{b}\right\| - |p-p_0|\,|\beta|$$
$$> \left\|(p-p_0)\frac{a}{b}\right\| - L|\beta| = \left\|(p-p_0)\frac{a}{b}\right\| - \frac{1}{2b}.$$

Thus

$$\left\|(p-p_0)\frac{a}{b}\right\|<\frac{1}{2y}+\frac{1}{2b}\leqq\frac{1}{b},$$

whence

$$p\equiv p_0 \pmod b). \tag{35}$$

Therefore

$$\frac{1}{2y}>\|p\alpha-p_0\alpha\|=\left\|(p-p_0)\frac{a}{b}+(p-p_0)\beta\right\|=\|(p-p_0)\beta\|. \tag{36}$$

Since

$$|(p-p_0)\beta|<L|\beta|=\frac{1}{2b}\leqq\frac{1}{2},$$

it follows from (36) that

$$\frac{1}{2y}>|p-p_0|\,|\beta|,$$

hence

$$|p-p_0|<\frac{1}{2|\beta|y}.$$

Thus, either there are no primes p with $(j-1)L<p\leqq jL$, $(p,b)=1$ and $\frac{k-1}{2y}\leqq\{p\alpha\}<\frac{k}{2y}$, or for some p_0 we have

$$\begin{aligned}\sum_{\substack{(j-1)L<p\leqq jL\\(p,b)=1\\\frac{k-1}{2y}\leqq\{p\alpha\}<\frac{k}{2y}}}1&\leqq\sum_{\substack{p_0-\frac{1}{2|\beta|y}<p<p_0+\frac{1}{2|\beta|y}\\p\equiv p_0(\mathrm{mod}\,b)}}1\\&\leqq\pi\left(p_0+\frac{1}{2|\beta|y},b,p_0\right)-\pi\left(p_0-\frac{1}{2|\beta|y},b,p_0\right).\end{aligned} \tag{37}$$

By (32), $\frac{1}{|\beta|y}>\frac{bQ}{y}$. Thus, for $N>N_{11}$, the right hand side of inequality (37) is, by (18) and Lemma 7,

$$\leqq\frac{\frac{2}{|\beta|y}}{\phi(b)\log\left(\frac{1}{|\beta|yb}\right)}$$

and, by (33), is

$$\leqq\frac{4bL}{y\phi(b)\log\left(\frac{Q}{y}\right)}\leqq\frac{c_{24}bL}{y\phi(b)\log N}.$$

It now follows from (34), that

$$|S(\alpha)| \leqq \sum_{j=1}^{\left[\frac{4N}{Ly}\right]+1} \sum_{k=1}^{2y} \left(\min\left(y, \frac{1}{2\left\|\frac{k-1}{2y}\right\|} \right) + \min\left(y, \frac{1}{2\left\|\frac{k}{2y}\right\|} \right) \right) \frac{c_{24} b L}{y\phi(b)\log N}$$

$$\leqq \left(\left[\frac{4N}{Ly}\right]+1 \right) \frac{c_{25} b L}{y\phi(b)\log N} \sum_{k=0}^{y} \min\left(y, \frac{1}{2\left\|\frac{k}{2y}\right\|} \right)$$

$$\leqq c_{26} \frac{Nb}{y^2 \phi(b)\log N} \left(y + \sum_{k=1}^{y} \frac{y}{k} \right) \leqq c_{27} \frac{N \log y b}{y \log N \phi(b)}$$

which, by Lemma 3, is

$$\leqq c_{28} \frac{N \log y \log\log b}{y \log N}.$$

Since $b \leqq y$ we have

$$(38) \qquad |S(\alpha)| \leqq \frac{c_{28} N \log y \log\log y}{y \log N}.$$

for $\alpha \in T_3$ provided that $N > N_{12}$.

Finally we assume that $\alpha \in T_4$. Put

$$L_1 = \min\left(\frac{N}{y}, \frac{1}{2|\beta|\, y} \right).$$

Then, by (30), for $N > N_9$,

$$(39) \qquad |S(\alpha)| \leqq \sum_{\substack{p \leqq \frac{4N}{y} \\ (p,b)=1}} \min\left(y, \frac{1}{2p\|\alpha\|} \right)$$

$$\leqq \sum_{j=1}^{\left[\frac{4N}{L_1 y}\right]+1} \sum_{\substack{(j-1)L_1 < p \leqq jL_1 \\ (p,b)=1}} \min\left(y, \frac{1}{2\|p\alpha\|} \right).$$

Now if $\frac{1}{2\|p\alpha\|} < y$ with $(j-1)L_1 < p \leqq jL_1$ and n is defined by $p \equiv n \pmod b$ with $jL_1 - b < n \leqq jL_1$ then

$$\|p\alpha\| = \left\| p\left(\frac{a}{b} + \beta \right) \right\| = \left\| \frac{an}{b} + n\beta + (p-n)\beta \right\|$$

$$\geqq \left\| \frac{1}{b}(an + nb\beta) \right\| - |p-n|\,|\beta|$$

and since $|p-n|\,|\beta| \leqq L_1 |\beta| \leqq \frac{1}{2y} < \|p\alpha\|$ we have

$$2\|p\alpha\| \geqq \left\| \frac{1}{b}(an+nb\beta) \right\|,$$

hence

$$\min\left(y, \frac{1}{2\|p\alpha\|}\right) \leqq \min\left(y, \frac{1}{\left\|\frac{1}{b}(an+nb\beta)\right\|}\right).$$

Therefore, by (39),

$$|S(\alpha)| \leqq \sum_{j=1}^{\left[\frac{4N}{L_1 y}\right]+1} \sum_{\substack{jL_1-b<n\leqq jL_1 \\ (n,b)=1}} \min\left(y, \frac{1}{\left\|\frac{1}{b}(an+nb\beta)\right\|}\right) \sum_{\substack{(j-1)L_1<p\leqq jL_1 \\ p\equiv n \pmod b}} 1. \tag{40}$$

We see from (25) and the fact that $b > y$ that $\frac{1}{2|\beta|\,y} > \frac{Q}{2}$ hence that $L_1 > \frac{Q}{2}$. For $N > N_5$ we have, by (17) and (26),

$$\frac{L_1}{b} > \frac{Q}{2b} \geqq \frac{N}{2y^6(\log N)^{20}} \geqq N^{3\varepsilon}, \tag{41}$$

whence, from Lemma 7,

$$\sum_{\substack{(j-1)L_1<p\leqq jL_1 \\ p\equiv n \pmod b}} 1 < \frac{2L_1}{\phi(b)\log\left(\frac{L_1}{b}\right)} \leqq \frac{L_1}{\varepsilon\phi(b)\log N}. \tag{42}$$

Combining (40) and (42) we obtain

$$|S(\alpha)| \leqq \frac{L_1}{\varepsilon\phi(b)\log N} \sum_{j=1}^{\left[\frac{4N}{L_1 y}\right]+1} \sum_{\substack{jL_1-b<n\leqq jL_1 \\ (n,b)=1}} \min\left(y, \frac{1}{\left\|\frac{1}{b}(an+nb\beta)\right\|}\right).$$

We may estimate the inner sum above by means of Lemma 6 with $h = jL_1 - b + 1$, $q = b$ and $\rho(n) = nb\beta$. Then, by (17) and (26),

$$\lambda = \max_{jL_1-b<n\leqq jL_1} nb\beta - \min_{jL_1-b<n\leqq jL_1} nb\beta$$

$$\leqq b^2|\beta| < \frac{b}{Q} < \frac{y^6(\log N)^{20}}{N} < 1,$$

for $N > N_5$. Thus

$$|S(\alpha)| \leqq \frac{L_1}{\varepsilon\phi(b)\log N}\left(\left[\frac{4N}{L_1 y}\right]+1\right) c_{12}\phi(b)\log y,$$

and, since $L_1 \leqq \frac{N}{y}$,

$$|S(\alpha)| \leqq \frac{c_{29} N \log y}{\varepsilon y \log N}. \tag{43}$$

If $y<\frac{N^{\frac{1}{12}}}{2(\log N)^4}$ then we may replace 3ε in (41) by $\frac{1}{2}$ and consequently ε in (43) by 1. On the other hand if $y\geqq\frac{N^{\frac{1}{12}}}{2(\log N)^4}$ then certainly $\frac{1}{\varepsilon}<\log\log y$ for $N>N_{13}$. Thus Lemma 11 holds for $\alpha\in T_4$ and, by (29), (31) and (38), the proof is complete.

Lemma 12. *If $N>N_{14}$ and n is an integer satisfying $\frac{30N}{y}<n\leqq 2N$ then*

$$v_n>c_{30}\frac{N}{y^2\log N}.$$

Proof. If n satisfies $\frac{30N}{y}<n\leqq 2N$ then for $N>N_{15}$,

$$v_n=\sum_{j=n-U+1}^{n} d_j=\sum_{\substack{n-U<mp\leqq n\\ m\leqq y\\ \frac{2N}{y}<p\leqq\frac{4N}{y}}} 1=\sum_{m\leqq y}\ \sum_{\max\left(\frac{n-U}{m},\frac{2N}{y}\right)<p\leqq\min\left(\frac{n}{m},\frac{4N}{y}\right)} 1.$$

Notice that if $m\leqq\frac{11}{30}\frac{ny}{N}$ then, by (16),

$$\frac{n-U}{m}\geqq\frac{30}{11}\frac{N}{y}-\frac{N}{y^2m}\geqq\frac{30}{11}\frac{N}{y}\left(1-\frac{11}{30y}\right)\geqq\frac{2N}{y}.$$

Further if $\frac{9}{30}\frac{ny}{N}<m$ then

$$\frac{n}{m}<\frac{30}{9}\frac{N}{y}<\frac{4N}{y}.$$

Since $\frac{11}{30}\frac{ny}{N}\leqq\frac{22}{30}y<y$ we conclude that

$$v_n>\sum_{\frac{9}{30}\frac{ny}{N}<m\leqq\frac{11}{30}\frac{ny}{N}}\ \sum_{\frac{n-U}{m}<p\leqq\frac{n}{m}} 1=\sum_{\frac{9}{30}\frac{ny}{N}<m\leqq\frac{11}{30}\frac{ny}{N}}\left(\pi\left(\frac{n}{m}\right)-\pi\left(\frac{n-U}{m}\right)\right).$$

We may now apply Lemma 8 with $x=\frac{n-U}{m}$ and $y=\frac{U}{m}$ since for $N>N_{16}$ and $m\leqq\frac{11}{30}\frac{ny}{N}$ we have, by (18),

$$\left(\frac{n-U}{m}\right)^{\frac{12}{21}}<\left(\frac{n}{m}\right)^{\frac{4}{7}}=\frac{(n^4m^3)^{\frac{1}{7}}}{m}\leqq\frac{n}{m}\left(\frac{11}{30}\frac{y}{N}\right)^{\frac{3}{7}}\leqq\frac{2N}{m}\left(\frac{11}{30}\frac{y}{N}\right)^{\frac{3}{7}}<\frac{U}{m}.$$

Thus, for $N > N_{17}$,

$$v_n > \sum_{\frac{9}{30}\frac{ny}{N} < m \leqq \frac{11}{30}\frac{ny}{N}} c_{31} \frac{U}{m \log\left(\frac{U}{m}\right)} > \frac{c_{31} U}{\log U} \sum_{\frac{9}{30}\frac{ny}{N} < m \leqq \frac{11}{30}\frac{ny}{N}} \frac{1}{m}$$

and, by (18), is

$$> c_{31} \frac{N}{y^2 \log N} \left(\frac{30}{11}\frac{N}{ny}\right)\left(\frac{2}{30}\frac{ny}{N} - 1\right) > c_{32} \frac{N}{y^2 \log N}\left(1 - \frac{15N}{ny}\right),$$

which, since $n > \frac{30N}{y}$, is

$$> c_{33} \frac{N}{y^2 \log N},$$

as required.

§ 4. Proof of the theorem

We shall first prove (4). We have for $N > N_{18}$,

$$\left| J - \frac{1}{U} \int_0^1 F(\alpha)\, G(\alpha)\, U(-\alpha)\, S(-\alpha)\, d\alpha \right|$$

$$= \left| \int_{-\delta}^{\delta} F(\alpha)\, G(\alpha)\, S(-\alpha) \left(1 - \frac{U(-\alpha)}{U}\right) d\alpha + \int_{\delta}^{1-\delta} F(\alpha)\, G(\alpha)\, S(-\alpha) \left(1 - \frac{U(-\alpha)}{U}\right) d\alpha \right|$$

$$\leqq \int_{-\delta}^{\delta} |F(\alpha)|\, |G(\alpha)|\, |S(-\alpha)|\, \frac{|U - U(-\alpha)|}{U}\, d\alpha$$

$$+ \int_{\delta}^{1-\delta} |F(\alpha)|\, |G(\alpha)|\, |S(-\alpha)| \left(1 + \left|\frac{U(-\alpha)}{U}\right|\right) d\alpha$$

which, by Lemma 2, is

$$\leqq \int_{-\delta}^{\delta} |F(\alpha)|\, |G(\alpha)|\, S\, \frac{4U^2 |\alpha|}{U}\, d\alpha + \int_{\delta}^{1-\delta} |F(\alpha)|\, |G(\alpha)| \left(\max_{\delta \leqq \beta \leqq 1-\delta} |S(\beta)|\right) 2\, d\alpha$$

by Lemmas 10 and 11, is

$$\leqq \int_{-\delta}^{\delta} |F(\alpha)|\, |G(\alpha)|\, \frac{40N}{\log N}\, U\delta\, d\alpha + \int_{\delta}^{1-\delta} |F(\alpha)|\, |G(\alpha)|\, 2c_{19} \frac{N \log y \log\log y}{y \log N}\, d\alpha$$

$$\leqq \left(\frac{5N}{y \log N} + 2c_{19} \frac{N \log y \log\log y}{y \log N}\right) \int_0^1 |F(\alpha)|\, |G(\alpha)|\, d\alpha$$

and by Cauchy's inequality and Parseval's formula is

$$\leqq c_{34} \frac{N \log y \log\log y}{y \log N} \left(\left(\int_0^1 |F(\alpha)|^2\, d\alpha\right)\left(\int_0^1 |G(\alpha)|^2\, d\alpha\right)\right)^{\frac{1}{2}},$$

and thus is

$$\leqq c_{34} \frac{N \log y \log\log y}{y \log N} (|A|\, |B|)^{\frac{1}{2}}. \tag{44}$$

Furthermore,

$$I=\int_0^1 F(\alpha)\,G(\alpha)\,U(-\alpha)\,S(-\alpha)\,d\alpha=\int_0^1\left(\sum_{n=1}^{2N} h_n e(n\alpha)\right)\left(\sum_{n=1}^{4N+U-1} v_n e(-n\alpha)\right)d\alpha$$
$$=\sum_{n=1}^{2N} h_n v_n.$$

Since h_n and v_n are non-negative for $n=1,\dots,2N$,

$$I\geqq\sum_{\frac{30N}{y}<n\leqq 2N} h_n v_n,$$

and by Lemma 12,

$$I\geqq\frac{c_{30}N}{y^2\log N}\sum_{\frac{30N}{y}<n\leqq 2N} h_n=\frac{c_{30}N}{y^2\log N}\sum_{\substack{a\in A,\,b\in B\\ \frac{30N}{y}<a+b\leqq 2N}} 1.$$

Observe that $\frac{30N}{y}\leqq\frac{3(|A|\,|B|)^{\frac{1}{2}}}{10}\leqq\frac{3}{10}\max(|A|,|B|)$ and thus

$$\sum_{\substack{a\in A,\,b\in B\\ \frac{30N}{y}<a+b\leqq 2N}} 1\geqq\frac{7}{10}|A|\,|B|.$$

Therefore

$$I\geqq\frac{c_{35}N|A|\,|B|}{y^2\log N}. \tag{45}$$

It follows from (44) and (45) that

$$|J|\geqq\frac{|I|}{U}-c_{34}\frac{N\log y\,\log\log y}{y\log N}(|A|\,|B|)^{\frac{1}{2}}$$
$$\geqq\frac{c_{35}}{\log N}\left(|A|\,|B|\;-c_{36}\frac{N\log y\,\log\log y}{y}(|A|\,|B|)^{\frac{1}{2}}\right).$$

Since $y=\omega R\log R\,\log\log R<\omega R^3$ we have

$$|J|\geqq\frac{c_{35}|A|\,|B|}{\log N}\left(1-c_{36}\frac{\log\omega R^3\,\log\log\omega R^3}{3\omega\log R\,\log\log R}\right)$$
$$\geqq\frac{c_{35}|A|\,|B|}{\log N}\left(1-c_{37}\frac{\log\omega\,\log\log\omega}{\omega}\right).$$

On choosing ω sufficiently large we find that

$$|J|\geqq\frac{c_{35}|A|\,|B|}{2\log N}.$$

Since J is non-negative, (20) holds and this completes the proof of (4).

The proof of (5) is essentially the same as that of (4). First observe that we may assume, without loss of generality, that $|A| \geqq |B|$. Next define z to be the smallest positive integer for which

$$|A \cap \{1, \ldots, z\}| \geqq \frac{|A|}{2}.$$

Put

$$A_1 = A \cap \left\{1, \ldots, z - \left[\frac{3|A|}{10}\right] - 1\right\},$$

$$A_2 = A \cap \left\{z + \left[\frac{3|A|}{10}\right] + 1, \ldots, N\right\},$$

$$B_1 = B \cap \{1, \ldots, z\},$$

$$B_2 = B \cap \{z+1, \ldots, N\}.$$

Note that the minimum of $|A_1|$ and $|A_2|$ is at least $\frac{|A|}{10}$ for $N > N_{19}$. Further the maximum of $|B_1|$ and $|B_2|$ is at least $\frac{|B|}{2}$.

We shall consider separately the case $|B_1| \geqq |B_2|$ and the case $|B_1| < |B_2|$. Assume first that $|B_1| \geqq |B_2|$. Then, since $\frac{30N}{y} \leqq \frac{3}{10}|A|$,

$$\sum_{\substack{a \in A,\, b \in B \\ \frac{30N}{y} < a - b \leqq N}} 1 \geqq |A_2|\,|B_1| \geqq \frac{|A|\,|B|}{20}. \tag{46}$$

We now replace $G(\alpha)$ by $G(-\alpha)$ in the above argument. Then

$$F(\alpha)\,G(-\alpha) = \sum_{a \in A,\, b \in B} e\big((a-b)\,\alpha\big) = \sum_{n=-N}^{N} h'_n e(n\alpha),$$

where

$$h'_n = \sum_{\substack{a \in A,\, b \in B \\ a - b = n}} 1.$$

Further we put $J' = \int_0^1 F(\alpha)\,G(-\alpha)\,S(-\alpha)\,d\alpha$ and $I' = \int_0^1 F(\alpha)\,G(-\alpha)\,U(-\alpha)\,S(-\alpha)\,d\alpha$. As before it suffices to show that $J' > \frac{c_3|A|\,|B|}{\log N}$. We have $I' = \sum_{n=1}^{N} h'_n v_n$ hence, by Lemma 12,

$$I' \geqq \frac{c_{38}N}{y^2 \log N} \sum_{\frac{30N}{y} < n \leqq N} h'_n.$$

Thus, by (46),

$$I' \geqq c_{39} \frac{N|A|\,|B|}{y^2 \log N}, \tag{47}$$

and the required lower bound for J' follows as above.

Finally we consider the case $|B_1|<|B_2|$. In this case

$$\sum_{\substack{a\in A,\, b\in B\\ \frac{30N}{y}<b-a\leq N}} 1 \geqq |A_1|\,|B_2| \geqq \frac{|A|\,|B|}{20}. \tag{48}$$

We then put

$$F(-\alpha)\,G(\alpha)=\sum_{a\in A,\, b\in B} e\big((b-a)\,\alpha\big)=\sum_{n=-N}^{N} h_n'' e(n\alpha),$$

where

$$h_n''=\sum_{\substack{a\in A,\, b\in B\\ b-a=n}} 1.$$

Further we put $J''=\int_0^1 F(-\alpha)\,G(\alpha)\,S(-\alpha)\,d\alpha$ and $I''=\int_0^1 F(-\alpha)\,G(\alpha)\,U(-\alpha)\,S(-\alpha)\,d\alpha$. Employing Lemma 12 and (48) we find that (47) holds with I' replaced by I''. We may now argue as above to show that $J''>c_3\dfrac{|A|\,|B|}{\log N}$. Since $P(b-a)=P(a-b)$ our result now follows.

Remark. By modifying the proof of Lemma 11 it is possible to show that there exists an effectively computable positive absolute constant c such that if N is a positive integer and y is a real number with $3\leqq y\leqq N^{\frac{1}{6}}$ then

$$\sum_{p\leqq N} \min(y,\, \|p\alpha\|^{-1}) < c\,\frac{N\log y\log\log y}{\log N},$$

for all real numbers α with $N^{-1}\leqq\alpha\leqq 1-N^{-1}$; the summation above is over primes p with $p\leqq N$. We shall establish this result in a subsequent paper.

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Eingegangen 6. Juni 1985