Pure Mathematics
Analysis Comprehensive Exam

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This exam has 8 questions; attempt all of them. Point values are given. Justify all of your answers and clearly indicate any major theorems which are used in your proofs or calculations.

[12] 1. (a) Let $\mathbb{R}$ be viewed as a vector space over $\mathbb{Q}$. By using Zorn's Lemma, prove that there exists a basis B of $\mathbb{R}$ over $\mathbb{Q}$ such that $1, \sqrt{2} \in B$.

(b) Prove that the basis B found in part (a) has to be an infinite uncountable set.

For the remainder of the question, we will use the following definition: a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *additive* when it has the property that

$$f(s+t) = f(s) + f(t), \quad \forall\, s, t \in \mathbb{R}.$$

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function. Prove that $f(qt) = qf(t)$ for all $q \in \mathbb{Q}$ and $t \in \mathbb{R}$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that $f(1) = 2$. Suppose moreover that f is continuous at $\sqrt{2}$. Prove that $f(\sqrt{2}) = 2\sqrt{2}$.

(e) Does there exist an additive function $g : \mathbb{R} \to \mathbb{R}$ (not required to satisfy any continuity conditions) such that $g(1) = 2$ and $g(\sqrt{2}) = 3$?

[10] 2. Let $X = [0,1]^{[0,1]} := \{f \mid f : [0,1] \to [0,1]\}$, be endowed with the topology $\mathcal{T}$ of pointwise convergence.

(a) Write a basis of open sets for the topological space $(X, \mathcal{T})$. Explain why this is a compact Hausdorff space.

(b) For every $n \in \mathbb{N}$, consider the function $f_n \in X$ defined by the following formula:

$$f_n(t) = 10^n t - \lfloor 10^n t \rfloor, \quad \forall\, t \in [0,1]$$

(where $\lfloor s \rfloor \in \mathbb{Z}$ denotes the "floor", or "integer part" of a number $s \in \mathbb{R}$). Prove the following: it is not possible to find some indices $n(1) < n(2) < \cdots < n(k) < \cdots$ in $\mathbb{N}$ such that the sequence $(f_{n(k)})_{k=1}^{\infty}$ is pointwise convergent.

(c) Is the space $(X, \mathcal{T})$ metrizable?

[10] 3. Let $(f_n)_{n=1}^{\infty}$ be a sequence of Lebesgue integrable functions from $[0,1]$ to $\mathbb{R}$ which satisfy $\int_0^1 |f_n(x)|\,dx \le 1$ for each n.

(a) Show, for each n, that

$$g_n(t) = \int_0^1 \sqrt{1+t+x}\, f_n(x)\,dx$$

defines a continuous function on $[0,1]$.

(b) Prove the following: there exist $n(1) < n(2) < \cdots < n(k) < \cdots$ in $\mathbb{N}$ and a continuous function $g : [0,1] \to \mathbb{R}$ such that $\lim_{k\to\infty} g_{n(k)} = g$ uniformly on $[0,1]$.

[15] 4. Let a parameter $0 < \alpha \le 1$ be given. We construct a Cantor-type set $C_\alpha \subset [0,1]$ as follows.

Let $C_1 = [0,1]$. Inductively, C_n is a disjoint union of 2^{n-1} pairwise disjoint closed intervals. We obtain C_{n+1} from C_n by removing the open middle interval, of length $\frac{\alpha}{3^n}$, from each of the constituent intervals of C_n. Let $C_\alpha = \bigcap_{n=1}^{\infty} C_n$.

(a) Explain why C_α is a non-empty, nowhere dense, compact set.

(b) Compute the Lebesgue measure, $m(C_\alpha)$.

(c) Show that there exists a continuous surjection, $\varphi : C_\alpha \to [0,1]$.

(d) Can φ, above, be arranged to be both continuous and invertible?

[15] 5. For $n \in \mathbb{Z}$, let $e_n : [-\pi,\pi] \to \mathbb{C}$ be defined by $e_n(t) = e^{int}$. We let $C[-\pi,\pi]$ denote the space of continuous $\mathbb{C}$-valued functions on the interval $[-\pi,\pi]$.

(a) Explain why the $\mathbb{C}$-linear span of the functions $\{e_n : n \in \mathbb{Z}\}$ is uniformly dense in the space $\{f \in C[-\pi,\pi] : f(\pi) = f(-\pi)\}$.

(b) Explain why the set $\{\frac{1}{\sqrt{2\pi}} e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for the space $L^2[-\pi,\pi]$ of square integrable functions (with respect to Lebesgue measure).

(c) Compute the inner product $\langle f, \frac{1}{\sqrt{2\pi}} e_n\rangle$, for f given by $f(t) := \cosh(t) = \frac{1}{2}(e^t + e^{-t})$, $-\pi \le t \le \pi$.

(d) Compute the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(1+n^2)^2}$.

(e) Compute the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$.

[12] 6. (a) Let

$$D := \{z \in \mathbb{C} : |z| < 1\} \text{ and } T := \{z \in \mathbb{C} : |z| = 1\}.$$

Let $f : \Omega \to \mathbb{C}$ be analytic on an open set $\Omega \supset D \cup T$, and suppose moreover that $f(z) \neq 0, \quad \forall\, z \in T$. Let $N_D(f)$ be the number of zeroes of f in D, counted with multiplicities. Indicate a closed path $\gamma : [0,1] \to \mathbb{C}$ for which $N_D(f)$ is equal to the winding number of γ around 0.

(b) Let D, T, Ω be as in (a), and let $f, g : \Omega \to \mathbb{C}$ be analytic and satisfy that

$$|f(z) + g(z)| < |f(z)| + |g(z)|, \quad \forall\, z \in T.$$

Fix a $z_o \in T$, and let S denote the closed line segment in $\mathbb{C}$ which has endpoints at $f(z_o)$ and at $-g(z_o)$. Prove that $0 \notin S$.

(c) Let D, T, Ω and $f, g : \Omega \to \mathbb{C}$ be as in (b). Prove that $N_D(f) = N_D(g)$.

[15] 7. (a) Let $U = \mathbb{C} \backslash \{it : t \leq 0 \text{ in } \mathbb{R}\}$ and let $L : U \to \mathbb{C}$ be the branch of logarithm given by

$$L(z) = \int_\gamma \frac{dw}{w}, \quad z \in U,$$

where $\gamma : [a,b] \to U$ is any piecewise smooth curve with $\gamma(a) = 1$ and $\gamma(b) = z$. Show that for $r > 0$ and $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ we have

$$L(re^{i\theta}) = \log r + i\theta.$$

(b) Given $0 < s < 1$ calculate the value of the improper Riemann integral

$$H(s) = \int_0^\infty \frac{t^s}{1+t^2}\, dt.$$

(c) Explain why, or why not, the following derivative formula makes sense:

$$H'(s) = \int_0^\infty \frac{t^s \log t}{1+t^2}\, dt.$$

[12] 8. Given a non-empty set X and a transformation $T : X \to X$, we denote by $T^{(n)} : X \to X$ the transformation which is obtained by composing T with itself n times. An element $x \in X$ is said to be *periodic for* T when there exists $p_x \in \mathbb{N}$ such that $T^{(p_x)}(x) = x$. For such an $x \in X$, the number p_x is called a *period of* x *under the transformation* T.

(a) Let X be a normed vector space over $\mathbb{R}$ and let Y be a closed linear subspace of X, where $Y \neq X$. Prove that Y is nowhere dense in X.

(b) Let X be a Banach space over $\mathbb{R}$, and let $T : X \to X$ be a continuous linear operator for which every x in X is periodic for T. Prove that there exists a $p_o \in \mathbb{N}$ which is a common period under T for *all* the vectors in X (i.e. $T^{(p_o)}(x) = x$ for all x in X).

(c) Show that for the sequence space c_{oo}, consisting of sequences of real numbers $x = (x_1, x_2, \dots)$ which are 0 for all but finitely many entries x_k, that there is a linear operator $T : c_{oo} \to c_{oo}$ for which

- every x in c_{oo} is periodic for T,
- T is continuous when c_{oo} is given the norm $\|x\|_\infty = \max_{k=1,2,\dots} |x_k|$; and
- there is no common period p_o for all x in c_{oo}.

Hence the assumption of completeness for X is essential for part (b), above.