Pure Mathematics Measure Theory and Fourier Analysis Qualifying Examination
University of Waterloo
September 16, 2022

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.
2. Check for questions on both sides of each page.
3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.
4. Do not write on the Crowdmark QR code at the top of each page.
5. Use a dark pencil or pen for your work.
6. All questions are equally weighted.

1. Let H be a (real or complex) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

   (a) Prove the *parallelogram law*:

   $$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \qquad (x, y \in H).$$

   (b) Let $K \subseteq H$ be a subspace. Define $K^\perp$, the orthogonal complement of K.

   (c) Let $K \subseteq H$ be a closed linear subspace and let $x \in H$. Prove that there is a unique element $Px \in K$ such that

   $$\|x - Px\| = \inf\{\|x-y\| : y \in K\}.$$

   (d) Prove that $x - Px \in K^\perp$ for each $x \in H$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. Consider the Hilbert space $L^2(\mathbb{T}) \cong L^2[-\pi,\pi]$ of 2π-periodic complex-valued square integrable measurable functions, equipped with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt \qquad (f, g \in L^2(\mathbb{T})).$$

Let $f \in L^2(\mathbb{T})$ satisfy

$$(n\hat{f}(n))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}),$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$ is the nth Fourier coefficient of f. Prove that the Fourier series of f converges uniformly and that f is a.e. equal to a differentiable function.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Let $f \in L^1(\mathbb{R})$, $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ a bounded and continuous function, and define

$$F : \mathbb{R} \to \mathbb{C}; \quad F(x) = \int_{\mathbb{R}} f(y)\psi(x,y)dy.$$

Assume that $\frac{\partial \psi}{\partial x}$ exists on $\mathbb{R} \times \mathbb{R}$, that there exists a non-negative measurable function g on $\mathbb{R}$ such that $fg \in L^1(\mathbb{R})$, and

$$\left| \frac{\partial \psi}{\partial x}(x,y) \right| \leq g(y) \qquad (\text{a.e. } y \in \mathbb{R}).$$

Prove that F is differentiable on $\mathbb{R}$ and

$$F'(x) = \int_{\mathbb{R}} f(y) \frac{\partial \psi}{\partial x}(x,y)dy \qquad (x \in \mathbb{R}).$$

Extra page for answers. Please specify the question number here and the use of this page on the question page.

4. *(a)* Let $A \subset \mathbb{R}$ be a Lebesgue measurable set with $m(A) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m(A \cap I) \geq \alpha m(I).$$

(b) Let $B \subset \mathbb{R}$ be a Lebesgue measurable set with $m(B) > 0$. Show that

$$B - B = \{x - y : \ x, y \in B\}$$

contains an (open) interval.

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